EVEN WEAKER CONSISTENCY FOR BOUNDED WIDTH CSP TEMPLATES

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Abstract. Kozik [3] has shown that every "pq-consistent" instance of a CSP template with bounded relational width has a solution. In an earlier paper by Barto and Kozik [1], a slightly stronger consistency notion referred to as "weak prague instances" was introduced, involving three conditions (P1), (P2), and (P3). It's easy to see that weak prague instances are pq-consistent, and that pq-consistent instances satisfy (P1) and (P3), but neither converse holds. Here we introduce an even weaker consistency condition than (P1) and (P3), and show that any instance of a CSP template with bounded relational width which satisfies this weaker consistency condition must have a solution as well. The main technical tool used is Zhuk's theory of strong subalgebras from [4].

1. Introduction

We want to understand the weakest consistency condition that implies the existence of a solution, for CSP templates of bounded width. We'll start by reviewing some of the consistency conditions which have come up in earlier work, as our consistency condition will be closely related to them.

Definition 1 (From [1])**.** An instance $\mathbf{X}$ of a CSP with variable domains $\mathbb{A}_x$ is called a *weak Prague instance* if it satisfies the following three conditions.

(P1) The instance $\mathbf{X}$ is arc-consistent, that is, each constraint relation $\mathbb{R} \le \prod_{x_i} \mathbb{A}_{x_i}$ is subdirect.

(P2) For every variable x, every set $A \subseteq \mathbb{A}_x$, and every cycle p from x to x, we have the implication
$$A + p = A \implies A - p = A.$$

(P3) For every variable x, every set $A \subseteq \mathbb{A}_x$, and every pair of cycles p, q from x to x, we have the implication
$$A + p + q = A \implies A + p = A.$$

Definition 2. An instance $\mathbf{X}$ of a CSP with variable domains $\mathbb{A}_x$ is *cycle-consistent* if it is arc-consistent (P1), and satisfies the following condition.

(C) For every variable x, every $a \in \mathbb{A}_x$, and every cycle p from x to x, we have
$$a \in \{a\} + p.$$

Both cycle-consistency and weak Prague instances are strong enough to imply the existence of solutions in CSP templates of bounded relational width. However, neither one of these consistency conditions implies the other. A common weakening was introduced by Kozik [3].

Definition 3. An instance $\mathbf{X}$ of a CSP with variable domains $\mathbb{A}_x$ is *pq-consistent* if it is arc-consistent (P1), and satisfies the following condition.

(PQ) For every variable x, every $a \in \mathbb{A}_x$, and every pair of cycles p, q from x to x, there exists some $j \ge 0$ such that we have
$$a \in \{a\} + j(p + q) + p.$$

Proposition 1. *We have the following implications between the consistency notions defined above:*

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- *every weak Prague instance is pq-consistent,*
- *cycle-consistency implies pq-consistency,*
- *pq-consistency implies conditions (P1) and (P3).*

Definition 4. Say that a CSP instance is *weakly consistent* if it is arc-consistent (P1), and satisfies the following condition:

(W) For every variable x with domain $\mathbb{A}_x$, every element $a \in \mathbb{A}_x$, and every pair of cycles p, q from x to x such that $a \in \{a\} + p + q$, there exists some $j \geq 0$ such that

$$a \in \{a\} + j(p+q) + p.$$

Proposition 2. *A CSP is weakly consistent if and only if it is arc-consistent (P1) and satisfies the following generalization of condition (W):*

(W*) *For every variable x with domain $\mathbb{A}_x$, every subset $B \subseteq \mathbb{A}_x$, and every pair of cycles p, q from x to x such that $B \subseteq B + p + q$, there exists some $j \geq 0$ such that*

$$B \cap (B + j(p+q) + p) \neq \emptyset.$$

Proposition 3. *Every CSP instance which satisfies conditions (P1) and (P3) is weakly consistent.*

Our main result, proved in the next section, says that every weakly consistent instance of a bounded width CSP template has a solution.

2. Strategic subalgebras and strategic consistency

First we recall Zhuk's special subalgebras from [4].

Definition 5. Suppose $\mathbb{A}$ is a finite idempotent algebra.

- We say that $\mathbb{B}$ *strongly absorbs* $\mathbb{A}$ if for every term t which depends on all its arguments, we have $t(\mathbb{A}, ..., \mathbb{A}, \mathbb{B}, \mathbb{A}, ..., \mathbb{A}) \subseteq \mathbb{B}$ for every possible position of $\mathbb{B}$.
- We say that $\mathbb{B}$ *absorbs* $\mathbb{A}$, written $\mathbb{B} \lhd \mathbb{A}$, if there is some idempotent term t such that $t(\mathbb{B}, ..., \mathbb{B}, \mathbb{A}, \mathbb{B}, ..., \mathbb{B}) \subseteq \mathbb{B}$ for every possible position of $\mathbb{A}$.
- We say that $\mathbb{B}$ *binary absorbs* $\mathbb{A}$ if there is a term t as above which has arity 2.
- We say that $\mathbb{B}$ *centrally absorbs* $\mathbb{A}$ if $\mathbb{B} \lhd \mathbb{A}$ and for all $a \in \mathbb{A} \setminus \mathbb{B}$, we have $(a, a) \notin \mathrm{Sg}\{(\mathbb{B} \times \{a\}) \cup (\{a\} \times \mathbb{B})\}$.
- We say that $\mathbb{B}$ is a *projective subalgebra* of $\mathbb{A}$ if for every term t, there is a coordinate i such that $t(\mathbb{A}, ..., \mathbb{A}, \mathbb{B}, \mathbb{A}, ..., \mathbb{A}) \subseteq \mathbb{B}$ when the $\mathbb{B}$ occurs in the ith input of t.
- We say that $\mathbb{B}$ is a *PC subalgebra* of $\mathbb{A}$ if there is a congruence $\theta \in \mathrm{Con}(\mathbb{A})$ such that $\mathbb{B}$ is a congruence class of θ, and $\mathbb{A}/\theta$ is a product of polynomially complete algebras which each have no proper binary absorbing subalgebra, no proper centrally absorbing subalgebra, and no proper projective subalgebra.

Proposition 4 (Zhuk [4])**.** *If $\mathbb{A}$ is a Taylor algebra, then every projective subalgebra of $\mathbb{A}$ is also a binary absorbing subalgebra of $\mathbb{A}$.*

The next tool is a technical trick which allows us to fold the binary absorbing case into the centrally absorbing case.

Definition 6. We say that an idempotent algebra $\mathbb{A}$ is *strongly prepared* if for any $\mathbb{B} \in HSP(\mathbb{A})$, every binary absorbing subalgebra $\mathbb{C} \lhd \mathbb{B}$ is also a strongly absorbing subalgebra of $\mathbb{B}$.

Proposition 5. *Every finite idempotent bounded width algebra has a strongly prepared reduct which also has bounded width.*

Proof. This follows from Lemma 2 of [2]. ☐

Proposition 6. *Every strongly absorbing subalgebra of an idempotent algebra $\mathbb{A}$ is also a centrally absorbing subalgebra.*

Theorem 1. *Suppose $\mathbb{A}$ is finite, idempotent, strongly prepared, and bounded width, and that $|\mathbb{A}| > 1$. Then $\mathbb{A}$ either has a proper centrally absorbing subalgebra or a proper PC subalgebra.*

Proof. This follows from Theorem 3.3 of [4]: we don't have to worry about p-affine quotients since $\mathbb{A}$ has bounded width, we don't have to worry about projective subalgebras since $\mathbb{A}$ is Taylor, and we don't have to worry about binary absorbing subalgebras since $\mathbb{A}$ is strongly prepared. The only remaining cases are the cases of a centrally absorbing subalgebra and a PC subalgebra. □

Definition 7. We say that a subalgebra $\mathbb{B}$ of $\mathbb{A}$ is *strategic*, written $\mathbb{B} \le_S \mathbb{A}$, if there is some sequence of subalgebras $\mathbb{A} = \mathbb{A}_0 \ge \mathbb{A}_1 \ge \cdots \ge \mathbb{A}_n = \mathbb{B}$ such that for each $i < n$, either

- $\mathbb{A}_{i+1}$ centrally absorbs $\mathbb{A}_i$, or
- $\mathbb{A}_{i+1}$ is a PC subalgebra of $\mathbb{A}_i$.

We say that an element $a \in \mathbb{A}$ is *strategic* if we have $\{a\} \le_S \mathbb{A}$.

Proposition 7 (Preimage property [4])**.** *If $\pi : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is a surjective homomorphism and $\mathbb{C} \le_S \mathbb{B}$, then $\pi^{-1}(\mathbb{C}) \le_S \mathbb{A}$.*

Proposition 8 (Projection for central subalgebras [4])**.** *If $\pi : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is a surjective homomorphism and if $\mathbb{C}$ is a centrally absorbing subalgebra of $\mathbb{A}$, then $\pi(\mathbb{C})$ is a centrally absorbing subalgebra of $\mathbb{B}$.*

Proposition 9 (Weak projection for PC subalgebras [4])**.** *If $\pi : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is a surjective homomorphism and $\mathbb{B}$ has no proper centrally absorbing subalgebra, and if $\mathbb{C}$ is a PC subalgebra of $\mathbb{A}$, then $\pi(\mathbb{C})$ is a PC subalgebra of $\mathbb{B}$.*

Remark 1. If we modified the definition of strategic subalgebras by also allowing $\mathbb{A}_{i+1}$ to be any subalgebra of $\mathbb{A}_i$ which *contains* a nonempty subalgebra $\mathbb{A}'_{i+1} \lhd \mathbb{A}_i$ which is simultaneously binary absorbing and centrally absorbing, then we would be able to show that for $\pi : \mathbb{A} \twoheadrightarrow \mathbb{B}$ a surjective homomorphism of Taylor algebras we have $\mathbb{C} \le_{S'} \mathbb{A} \implies \pi(\mathbb{C}) \le_{S'} \mathbb{B}$ (where $\le_{S'}$ refers to the expanded definition of strategic subalgebra). However, this would require more casework overall, as well as stronger variants of some of Zhuk's results from [4].

Theorem 2 (Intersection property [4])**.** *If $\mathbb{B}, \mathbb{C}$ are two strategic subalgebras of $\mathbb{A}$ with $\mathbb{B} \cap \mathbb{C} \ne \emptyset$, then $\mathbb{B} \cap \mathbb{C}$ is a strategic subalgebra of $\mathbb{B}$.*

Theorem 3 (Helly property)**.** *If $\mathbb{B}_1, ..., \mathbb{B}_n$ is a collection of strategic subalgebras of $\mathbb{A}$ such that $\mathbb{B}_i \cap \mathbb{B}_j \ne \emptyset$ for all i, j, then $\mathbb{B}_1 \cap \cdots \cap \mathbb{B}_n \ne \emptyset$.*

Proof. If each $\mathbb{B}_i$ is either a centrally absorbing subalgebra or a PC subalgebra of $\mathbb{A}_i$, then this follows from Theorem 3.7 of [4]. The general case then follows from the intersection property together with an inductive argument. □

Corollary 1 (No essential subdirect relations)**.** *If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is a subdirect relation, $\mathbb{B}_i \le_S \mathbb{A}_i$ for all i, and $\pi_{i,j}(\mathbb{R}) \cap (\mathbb{B}_i \times \mathbb{B}_j) \ne \emptyset$ for all i, j, then $\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n) \ne \emptyset$.*

Definition 8. Say that a CSP instance is *strategically consistent* if it is arc-consistent (P1), and satisfies the following condition:

(S) For every variable x with domain $\mathbb{A}_x$, every strategic subalgebra $\mathbb{B} \le_S \mathbb{A}_x$, and every pair of cycles p, q from x to x such that $\mathbb{B} \subseteq \mathbb{B} + p + q$, there exists some $j \ge 0$ such that

$$\mathbb{B} \cap (\mathbb{B} + j(p+q) + p) \ne \emptyset.$$

Proposition 10. *Every weakly consistent CSP instance is strategically consistent.*

Lemma 1. *Suppose that a CSP instance is strategically consistent, with all variable domains finite idempotent strongly prepared bounded width algebras. Then if some variable domain has size greater than* 1*, there is some collection of strategic subalgebras of the variable domains, at least one of which is proper, such that reducing each variable domain to the corresponding strategic subalgebra produces an arc-consistent instance.*

Proof. Fix a variable x such that the corresponding domain $\mathbb{A}_x$ has more than one element, such that $\mathbb{A}_x$ has a proper centrally absorbing subalgebra if possible. We define a directed graph on the collection of proper nonempty subalgebras $\mathbb{B}$ of $\mathbb{A}_x$, with an edge $\mathbb{B} \to \mathbb{C}$ if there is some cycle p from x to x with $\mathbb{B}+p = \mathbb{C}$. If $\mathbb{A}_x$ has a proper centrally absorbing subalgebra, then we restrict our attention to the subdigraph of $\mathbb{B}$ which centrally absorb $\mathbb{A}_x$, and otherwise we restrict our attention to the subdigraph of $\mathbb{B}$ which are PC subalgebras of $\mathbb{A}_x$. Pick any $\mathbb{B}$ which is in a maximal strongly connected component of this restricted digraph.

Claim. For every cycle p from x to x, we have $\mathbb{B} \cap (\mathbb{B}+p) \neq \emptyset$.

Proof of claim. If $\mathbb{B}+p = \mathbb{A}_x$, then this is clear. Otherwise, $\mathbb{B}+p$ must be in the same strongly connected component of our restricted digraph as $\mathbb{B}$, so there must be some cycle q from x to x such that $\mathbb{B}+p+q = \mathbb{B}$. Since $\mathbb{B} \le_S \mathbb{A}_x$ (since $\mathbb{B}$ is in our restricted digraph), we can apply condition (S) to see that there must exist some $j \ge 0$ such that

$$\mathbb{B} \cap (\mathbb{B} + j(p+q) + p) \neq \emptyset.$$

Since $\mathbb{B} + j(p+q) + p = \mathbb{B} + p$, this proves the claim.

Now suppose that we restrict the domain of $\mathbb{A}_x$ to $\mathbb{B}$ and try to establish arc-consistency. If we ever reach a contradiction while trying to establish arc-consistency, then there is some proof-tree instance built out of relations of our CSP instance with some variable domains restricted from $\mathbb{A}_x$ to $\mathbb{B}$, which has no solutions. However, restricting any *two* variable domains from $\mathbb{A}_x$ to $\mathbb{B}$ gives us an instance with a solution (by the claim), so this situation would contradict Corollary 1.

Thus we must be able to eventually establish arc-consistency without shrinking any domain to the empty set. To finish, we apply the preimage property, the intersection property, and either the projection property for centrally absorbing subalgebras or the weak projection property for PC subalgebras to see that our reduced domains are all strategic subalgebras of the original domains. □

Lemma 2. *Suppose that a CSP instance* $\mathbf{X}$ *is strategically consistent, with all variable domains finite idempotent strongly prepared algebras. If* $\mathbf{X}'$ *is an arc-consistent reduced instance defined by replacing each variable domain of* $\mathbf{X}$ *with a strategic subalgebra, then* $\mathbf{X}'$ *is also strategically consistent.*

Proof. Let $\mathbb{A}_x$ be the variable domain of $\mathbf{X}$ corresponding to the variable x, and let $\mathbb{A}'_x$ be the corresponding variable domain in $\mathbf{X}'$. We will use $+$ to refer to adding cycles in $\mathbf{X}$, and $+'$ to refer to adding the corresponding cycles in $\mathbf{X}'$.

Claim. For any strategic subalgebra $\mathbb{B} \le_S \mathbb{A}'_x$ and for any cycle r from x to x such that $\mathbb{B} \cap (\mathbb{B}+r) \neq \emptyset$, we also have $\mathbb{B} \cap (\mathbb{B} +' r) \neq \emptyset$.

Proof of claim. Unravel the cycle r to get a path instance from one copy of x to another copy of x - call these copies x_1 and x_2. Let $\mathbb{R} \le_{sd} \mathbb{A}_{x_1} \times \cdots \times \mathbb{A}_{x_2}$ be the set of solutions to this path instance corresponding to r. By the preimage property, we see that

$$\mathbb{S}_1 = \pi_{x_1}^{-1}(\mathbb{B}) \text{ and } \mathbb{S}_2 = \pi_{x_2}^{-1}(\mathbb{B})$$

are strategic subalgebras of $\mathbb{R}$, and since $\mathbb{B} \cap (\mathbb{B}+r) \neq \emptyset$, the strategic subalgebras $\mathbb{S}_1, \mathbb{S}_2$ have nonempty intersection. Similarly, by the preimage property and the intersection property we see that

$$\mathbb{S}_3 = \mathbb{R} \cap (\mathbb{A}'_{x_1} \times \cdots \times \mathbb{A}'_{x_2})$$

is a strategic subalgebra of $\mathbb{R}$. Since $\mathbf{X}'$ is arc-consistent, we see that $\mathbb{S}_1 \cap \mathbb{S}_3 \neq \emptyset$ and $\mathbb{S}_2 \cap \mathbb{S}_3 \neq \emptyset$. Thus by the Helly property, we have $\mathbb{S}_1 \cap \mathbb{S}_2 \cap \mathbb{S}_3 \neq \emptyset$, that is, $\mathbb{B} \cap (\mathbb{B} +' r) \neq \emptyset$. We have proved the claim.

Now we can verify that $\mathbf{X}'$ satisfies condition (S). Suppose that $\mathbb{B}$ is any strategic subalgebra of $\mathbb{A}'_x$ such that there are cycles p, q from x to x with $\mathbb{B} \subseteq \mathbb{B} +' p +' q$. Then we also have $\mathbb{B} \subseteq \mathbb{B} + p + q$, so by (S) applied to the original instance $\mathbf{X}$, we see that there is some $j \geq 0$ such that

$$\mathbb{B} \cap (\mathbb{B} + j(p+q) + p) \neq \emptyset.$$

If we apply the claim to the cycle $r = j(p+q) + p$, we see that

$$\mathbb{B} \cap (\mathbb{B} +' j(p +' q) +' p) = \mathbb{B} \cap (\mathbb{B} +' r) \neq \emptyset,$$

so we have verified (S) for the reduced instance $\mathbf{X}'$ as well. □

Theorem 4. *Suppose that a CSP instance $\mathbf{X}$ is weakly consistent, with all variable domains finite idempotent bounded width algebras. Then the instance $\mathbf{X}$ has a solution.*

3. Exploration of weak consistency

One might wonder if it is possible to find a consistency notion which is weaker than *pq*-consistency, solves all bounded width CSP templates, and which only examines a single cycle at a time. The answer is no!

Example 1. Consider the following arc-consistent instance of 2-SAT with a single variable x, and two binary constraints: the first constraint says that $(x, x) \neq (0, 0)$, while the second constraint says that $(x, x) \neq (1, 1)$. Every cycle of this instance, considered as an instance with a number of variables equal to the length of the cycle, is a *pq*-consistent instance, but the full instance has no solution.

Let's see what (W) has to say about instances consisting of a single cycle - while keeping in mind that this is a very incomplete picture of what (W) means in general.

Proposition 11. *Suppose that $\mathbf{X}$ is a weakly consistent instance and p is a cycle from the variable x to x. Then the corresponding binary relation $\mathbb{P}_p \leq \mathbb{A}_x \times \mathbb{A}_x$ has the following properties.*

- *If $\theta \in \mathrm{Con}(\mathbb{A}_x)$ is the linking congruence of $\mathbb{P}_p$ on the first coordinate, then $\mathbb{P}_p \subseteq \theta$.*
- *If we define a digraph by $\mathbf{P} = (\mathbb{A}, \mathbb{P}_p)$, then every nontrivial strongly connected component of $\mathbf{P}$ has algebraic length 1.*

Problem 1. If an instance $\mathbf{X}$ is weakly consistent but not compatible with the algebraic structure, is its closure $\mathrm{Sg}(\mathbf{X})$ automatically weakly consistent? How about instances which satisfy (P1) and (P3)?

One possible further weakening of the condition (W) is the following:

(*) For every variable x with domain $\mathbb{A}_x$, every subset $B \subseteq \mathbb{A}_x$, and every pair of cycles p, q from x to x, we have

$$B = B + p + q \implies B \cap (B + p) \neq \emptyset.$$

Problem 2. Does every instance which satisfies conditions (P1) and (*) have a solution?

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