Stable subalgebras and weak consistency

Zarathustra Brady

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- A *variable* is a variable name x together with a variable domain $\mathbb{A}_x \in \mathcal{V}$.

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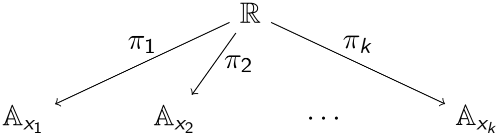
- 

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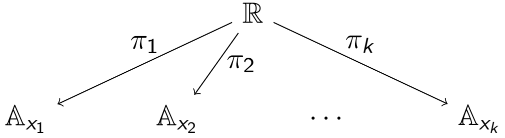
  - 

- A *solution* is an assignment $x \mapsto a_x \in \mathbb{A}_x$, such that for each constraint, $\exists r \in \mathbb{R}$ with

$$\pi_i(r) = a_{x_i}$$

for $i = 1, \ldots, k$.

Paths

- A *step* from y to z is a constraint

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and a pair i, j such that $x_i = y$ and $x_j = z$.

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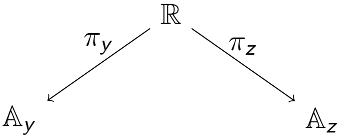
- 

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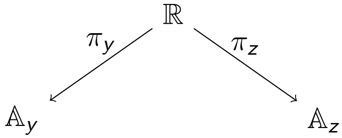
- 

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$$\mathbb{A}_y \xleftarrow{\ \pi_y\ } \mathbb{R} \xrightarrow{\ \pi_z\ } \mathbb{A}_z$$

- A *path* is a sequence of steps where the endpoints match up.
- We use additive notation for combining paths: $p + q$ means "first follow p, then q".

Propagating information along paths

- If $B \subseteq \mathbb{A}_y$ and p is a step from y to z through a relation $\mathbb{R}$, we write

$$B + p = B + \pi_{yz}(\mathbb{R}) = \pi_z(\pi_y^{-1}(B)) \subseteq \mathbb{A}_z.$$

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- If $\mathbb{B} \leq \mathbb{A}_y$ is a subalgebra, then $\mathbb{B} + p \leq \mathbb{A}_z$ is also a subalgebra.

Consistency

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- Beginner Sudoku players start by establishing arc-consistency, then they move on to establishing cycle-consistency.

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Theorem (Bulatov, Barto, Kozik)

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- $\mathcal{V}$ *is congruence meet-semidistributive,*
- *every cycle-consistent instance of* $\mathrm{CSP}(\mathcal{V})$ *has a solution.*

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- Before *pq*-consistency was introduced, there were "Prague instances".

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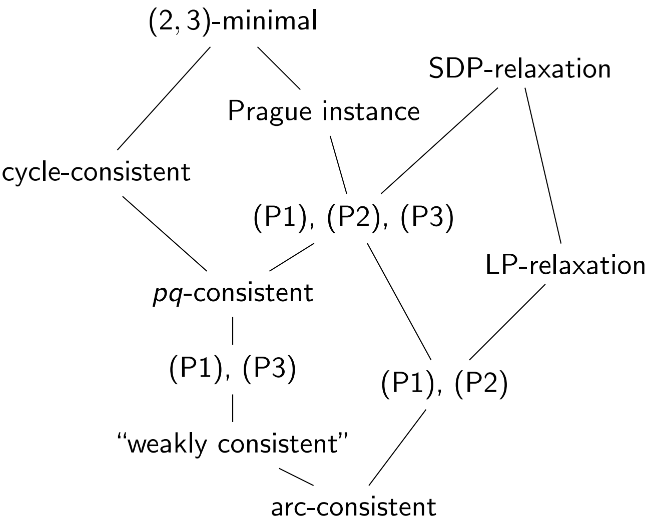
- Condition (P2) is closely related to the Linear Programming relaxation of the instance.
- Condition (P3) is closely related to the Semidefinite Programming relaxation of the instance.
- Barto asks: are (P1) and (P3) enough to guarantee solvability for bounded width CSPs?

Relationships between consistency notions



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- I call an instance *weakly consistent* if it satisfies:

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- My main result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite $\mathrm{SD}(\wedge)$ algebras, then every weakly consistent instance of $\mathrm{CSP}(\mathcal{V})$ has a solution.

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- The only algebraic tool needed to prove this result is the concept of a *stable subalgebra*, based on the ideas in Zhuk's paper "Strong subalgebras and the Constraint Satisfaction Problem".

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- Stable subalgebras are like absorbing subalgebras, but they are aimed at constraining the structure of *subdirect* relations instead of arbitrary relations.
- My definition of stable subalgebras is ugly, so instead I will describe the axioms that stable subalgebras satisfy.

Axioms for Stability

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  - (Pullback) if $\mathbb{D} \prec \mathbb{B}$, then $f^{-1}(\mathbb{D}) \prec \mathbb{A}$.
- (Helly) If $\mathbb{B}, \mathbb{C}, \mathbb{D} \prec \mathbb{A}$ have $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, $\mathbb{C} \cap \mathbb{D} \neq \emptyset$, and $\mathbb{B} \cap \mathbb{D} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} \neq \emptyset$.

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- (Ubiquity) For all $\mathbb{A} \in \mathcal{V}$, there is some $a \in \mathbb{A}$ such that $\{a\} \prec \mathbb{A}$.

Alternate forms of the axioms

- The propagation axiom is equivalent to:

$$\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B},\ \ \mathbb{C} \prec \mathbb{A} \quad \Longrightarrow \quad \mathbb{C} + \mathbb{R} \prec \mathbb{B}.$$

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- Modulo the intersection axiom, the Helly axiom is equivalent to:

$$\mathbb{B}_1, \ldots, \mathbb{B}_n \prec \mathbb{A}, \;\; \mathbb{B}_i \cap \mathbb{B}_j \neq \emptyset \;\; \forall i, j \quad \Longrightarrow \quad \bigcap_i \mathbb{B}_i \neq \emptyset.$$

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- The propagation, intersection, and Helly axioms imply that

$$\left.\begin{array}{l} \mathbb{R} \leq_{sd} \prod_i \mathbb{A}_i \\ \mathbb{B}_i \prec \mathbb{A}_i \\ \pi_{ij}(\mathbb{R}) \cap (\mathbb{B}_i \times \mathbb{B}_j) \neq \emptyset \end{array}\right\} \Longrightarrow \mathbb{R} \cap \left(\prod_i \mathbb{B}_i\right) \neq \emptyset.$$

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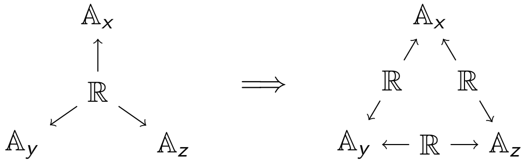

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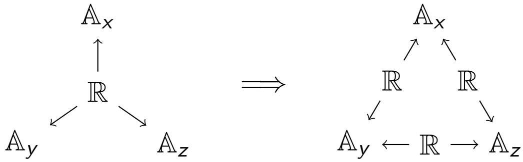

- If X is arc-consistent and X^{bin} has a *stable* solution, then this solution will also be a solution to X by the Helly axiom.

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- Main technical result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite idempotent $\mathrm{SD}(\wedge)$ *algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

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  - every element of a polynomially complete, absorption-free algebra is stable, and

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If $\mathcal{V}$ is a pseudovariety of finite idempotent SD($\wedge$) *algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

- My proof is an ad-hoc mess. I use König's Lemma to reduce to the case where $\mathcal{V}$ is finitely generated, take a convenient reduct...

- Morally, stability is generated by three basic cases:
  - Zhuk's "central" absorbing subalgebras are stable subalgebras,
  - every element of a polynomially complete, absorption-free algebra is stable, and
  - any subalgebra which contains a strongly absorbing subalgebra is stable.

Applying stability

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- Say an instance is *stably consistent* if:

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(S) if $\mathbb{B} \prec \mathbb{A}_x$ and $\mathbb{B} + p + q = \mathbb{B}$, then $\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset$.

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- We will prove that every stably consistent instance has a stable solution by induction.

Standard inductive strategy

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Step 1 produce an arc-consistent reduction with nice algebraic properties,

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- By a *reduction*, I mean replace all of the variable domains and constraint relations of the instance by subalgebras of the original ones.

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- We now try to restrict $\mathbb{A}_x$ to $\mathbb{B}$ for every $(x, \mathbb{B})$ in our maximal strongly connected component $\mathcal{C}$.

Step 1, continued

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- Looks good so far, but is this strong enough to guarantee arc-consistency?

Step 1, completed

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- By the Helly axiom, we can restrict all copies of $\mathbb{A}_x$ to $\mathbb{B}$ simultaneously.

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- We just need to show that

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- Unroll the path p (duplicating vertices that occur along it multiple times):

$$\begin{array}{ccccccccccccc}
 & & \mathbb{A}_u & & & & & & & & \mathbb{A}_v & & \\
 & & \twoheaduparrow & & & & & & & & \twoheaduparrow & & \\
\mathbb{A}_x & \twoheadleftarrow & \mathbb{R}_1 & \twoheadrightarrow & \mathbb{A}_y & \twoheadleftarrow & \mathbb{R}_2 & \twoheadrightarrow & \mathbb{A}_z & \twoheadleftarrow & \mathbb{R}_3 & \twoheadrightarrow & \mathbb{A}_x \\
 & & & & & & & & & & \twoheaddownarrow & & \\
 & & & & & & & & & & \mathbb{A}_w & &
\end{array}$$

Step 2, continued

- We need to show that there is a solution to the reduced path instance

$$\begin{array}{ccccccccccccc} & & \mathbb{A}'_u & & & & & & \mathbb{A}'_v & & \\ & & \twoheaduparrow & & & & & & \twoheaduparrow & & \\ \mathbb{A}'_x & \twoheadleftarrow & \mathbb{R}'_1 & \twoheadrightarrow & \mathbb{A}'_y & \twoheadleftarrow & \mathbb{R}'_2 & \twoheadrightarrow & \mathbb{A}'_z & \twoheadleftarrow & \mathbb{R}'_3 & \twoheadrightarrow & \mathbb{A}'_x \\ & & & & & & & & & & \twoheaddownarrow & & \\ & & & & & & & & & & \mathbb{A}'_w & & \end{array}$$

where the two copies of x are assigned values in $\mathbb{B}$.

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- Apply the Helly axiom to $\mathbb{S}_1, \mathbb{S}_2, \mathbb{R}' \prec \mathbb{R}$.

Applications to height one identities

- We can give a new characterization of locally finite SD($\wedge$) varieties:

Theorem (Z.)

If $\mathcal{V}$ is a locally finite variety, then $\mathcal{V}$ is SD($\wedge$) *if and only if there is a 4-ary term t which satisfies the identities*

$$t(x,x,y,z) \approx t(y,z,z,x) \approx t(z,x,y,x)$$

and

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- This can be proved by combining the weak consistency result with a Ramsey-theoretic argument.

Another height one identity

- A tougher application:

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If $\mathcal{V}$ is a locally finite SD($\wedge$) *variety, then $\mathcal{V}$ has a* 5*-ary "almost cyclic" term c which satisfies the identity*

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- This easily implies that every algebra in $\mathcal{V}$ of size ≤ 4 has a 5-ary cyclic term!

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- How much do we have to weaken the ubiquity axiom for stability concepts in pseudovarieties which are not SD($\wedge$)?
- Are there any CSPs which are solved by the Linear Programming relaxation, but which are not solved by enforcing (P1) and (P2)?

Thank you for your attention.