Rounding rules and vague solutions to bounded width CSPs

Zarathustra Brady

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- If $c = (x_1, ..., x_k) \in C_i$ is a constraint, then a solution a must satisfy $(a(x_1), ..., a(x_k)) \in R_i$.

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- An approximate solution with value 1 is the same thing as an ordinary solution.

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Theorem (Håstad)

If $\mathbf{A} = (\mathbb{Z}/p, \{x + y = z\}, ..., \{x + y = z + p - 1\})$*, then it is NP-hard to find an approximate solution* $a : X \to A$ *of value* $\frac{1}{p} + \epsilon$*, even if the instance* **X** *is promised to have value* $1 - \epsilon$*.*

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- Furthermore, Barto and Kozik's algorithm has

$$f(\epsilon) \ll \frac{\log\log(1/\epsilon)}{\log(1/\epsilon)}.$$

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- $\mathrm{Var}(\mathbb{A})$ *is congruence meet-semidistributive,*
- *every cycle-consistent instance of* $\mathrm{CSP}(\mathbf{A})$ *has a solution.*

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- We can define *approximate fractional solutions* similarly, with $r_i : C_i \to \Delta(A^k)$ instead of $r_i : C_i \to \Delta(R_i)$.

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- We say that the LP rounding scheme s *solves* $\mathrm{CSP}(\mathbf{A})$ if for every instance $\mathbf{X}$, and for every fractional solution

$$a : X \to \Delta(A), \quad r_i : C_i \to \Delta(R_i),$$

  the map

$$s \circ a : X \to A$$

  defines a homomorphism $\mathbf{X} \to \mathbf{A}$.

Example of an LP rounding scheme

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- For every n, the symmetric function s_n given by

$$s_n(x_1, \dots, x_n) = \begin{cases} +1 & \sum_i x_i > 0, \\ 0 & \sum_i x_i = 0, \\ -1 & \sum_i x_i < 0 \end{cases}$$

  is a polymorphism of $\mathbf{A}$.

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- Unfortunately, not every bounded width CSP has an LP rounding scheme:

$$\text{2-SAT} = (\{0,1\}, \{x \neq y\}, \{x \geq y\})$$

has no binary symmetric polymorphism.

From fractional solutions to preference relations

- If $p \in \Delta(A)$ is a probability distribution over A, then we can define a *total preorder* $\preceq_p$ on the powerset $\mathcal{P}(A)$:

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- We want to outlaw this sort of preference relation.

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- In particular, if $R \subseteq A^k$ is a relation, and $r \in \mathcal{V}(R)$, then we can define the ith marginal of r to be

$$(\pi_i \circ \iota)_*(r) \in \mathcal{V}(A),$$

where $\iota : R \hookrightarrow A^k$ is the inclusion.

Marginals of vague elements

- The map $S \mapsto \mathcal{V}(S)$ defines a *functor*.
- If $f : S \to T$ and $v \in \mathcal{V}(S)$, we define $f_*(v) \in \mathcal{V}(T)$ by

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  where $\iota : R \hookrightarrow A^k$ is the inclusion.
- Note that $\iota_*(r)$ is a vague element of A^k with support contained in R.

Vague solutions, take one

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- But describing a vague element of R_i sounds very onerous. We will make a simpler (weaker) definition.

Vague solutions, take two

Definition

If $R \subseteq A_1 \times \cdots \times A_k$, then a collection of vague elements $v_i \in \mathcal{V}(A_i)$ *vaguely satisfies* the relation R if there exists a preorder $\preceq_r$ on the disjoint union

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- for each i, the restriction of $\preceq_r$ to $\mathcal{P}(A_i)$ is $\preceq_{v_i}$,
- for each i, j and each $U \subseteq A_i$, we have

  $$U \preceq_r U + \pi_{ij}(R \cap (S_1 \times \cdots \times S_k)),$$

  where the S_i are the supports of the vague elements v_i.

Vague rounding schemes

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- A *vague rounding scheme* is just a map
$$s : \mathcal{V}(A) \to A.$$
- We say that the vague rounding scheme s *solves* $\mathrm{CSP}(\mathbf{A})$ if for every instance $\mathbf{X}$, and for every vague solution
$$a : X \to \mathcal{V}(A)$$
such that $(a(x_1), \ldots, a(x_k))$ vaguely satisfies R_i for each constraint $c = (x_1, \ldots, x_k) \in C_i$, the map
$$s \circ a : X \to A$$
defines a homomorphism $\mathbf{X} \to \mathbf{A}$.

Main result

Theorem (Z.)

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For a finite relational structure $\mathbf{A}$*, TFAE:*

- $\mathbf{A}$ *has bounded width,*
- *there is a vague rounding scheme* $s : \mathcal{V}(A) \to A$ *which solves* $\mathrm{CSP}(\mathbf{A})$*,*
- *for every* n*, and for every vague element* $v \in \mathcal{V}(\{1, ..., n\})$*, there is an* n*-ary polymorphism* $s_v \in \mathrm{Pol}(\mathbf{A})$*, such that for all*

$$f : \{1, ..., n\} \to \{1, ..., m\}$$

the height 1 identity

$$s_v(x_{f(1)}, ..., x_{f(n)}) \approx s_{f_*(v)}(x_1, ..., x_m)$$

is satisfied.

Multisorted CSPs

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- A constraint c now consists of a tuple $(x_1, \ldots, x_k)$ of variables, together with a *constraint relation*

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- A *solution* is a map $x \mapsto a_x$ such that for each constraint c as above, we have

$$(a_{x_1}, ..., a_{x_k}) \in \mathbb{R}.$$

Paths

- A *step* from y to z is a constraint

$$((x_1, \ldots, x_k), \mathbb{R})$$

and a pair i, j such that $x_i = y$ and $x_j = z$.

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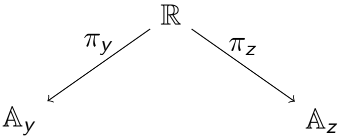

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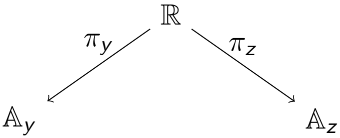

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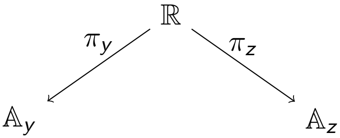

- A *path* is a sequence of steps where the endpoints match up.
- We use additive notation for combining paths: $p + q$ means "first follow p, then q".

Propagating information along paths

- If $B \subseteq \mathbb{A}_y$ and p is a step from y to z through a relation $\mathbb{R}$, we write

$$B + p = B + \pi_{yz}(\mathbb{R}) = \pi_z(\pi_y^{-1}(B) \cap \mathbb{R}) \subseteq \mathbb{A}_z.$$

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- If $\mathbb{B} \leq \mathbb{A}_y$ is a subalgebra, then $\mathbb{B} + p \leq \mathbb{A}_z$ is also a subalgebra.

Consistency

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- An instance is *cycle-consistent* if for all paths p from x to x, and for all $a \in \mathbb{A}_x$, we have
$$a \in \{a\} + p.$$
- Beginner Sudoku players start by establishing arc-consistency, then they move on to establishing cycle-consistency.

Weaker consistency!

- I call an instance *weakly consistent* if it satisfies:
  - (P1) arc-consistency, and
  - (W) $A + p + q = A$ implies $A \cap (A + p) \neq \emptyset$.

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- I will use this result, from a previous AAA conference:

Theorem (Z.)

If $\mathrm{Var}(\mathbb{A})$ *is* $\mathrm{SD}(\wedge)$*, then every weakly consistent instance of* $\mathrm{CSP}(\mathrm{Var}_{fin}(\mathbb{A}))$ *has a solution.*

Connection to vague solutions

Proposition

If an instance $\mathbf{X}$ *of a multisorted CSP is weakly consistent, then it has a vague solution*

$$x \mapsto a_x \in \mathcal{V}(\mathbb{A}_x)$$

such that each a_x *has support equal to* $\mathbb{A}_x$*.*

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- Extend $\preceq$ to a total preorder $\preceq'$ without changing the associated equivalence relation $\sim$.
- Let $\preceq_{a_x}$ be the restriction of $\preceq'$ to $\mathcal{P}(\mathbb{A}_x)$.

From a vague solution to a weakly consistent instance

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- The trick is to exploit the fact that everything is stated in terms of *total* preorders.

Compatibility between vague elements and functions

- If $f : \mathcal{P}(A) \to \mathbb{N}$ and $v \in \mathcal{V}(A)$, we say f is *compatible* with v if

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- If $f : \mathcal{P}(A_1) \sqcup \cdots \sqcup \mathcal{P}(A_k) \to \mathbb{N}$, and if $R \subseteq A_1 \times \cdots \times A_k$, we say f is *compatible* with R if
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- For $c = ((x_1, ..., x_k), \mathbb{R})$ and compatible $f : \mathcal{P}(\mathbb{A}_{x_1}) \sqcup \cdots \sqcup \mathcal{P}(\mathbb{A}_{x_k}) \to \mathbb{N}$, we introduce the constraint

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- By construction, if there is a path p from (x, f) to (x, f) in $\mathbf{X}_a^*$, and if $A \subseteq \mathbb{A}_x$, then

$$f(A) \le f(A + p), \quad \text{so} \quad A \preceq_{a_x} A + p.$$

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for all $(x, f) \in \mathbf{X}_a^*$ with $\mathrm{im}(f) \subseteq S$.
- If $a_{x_1}, \ldots, a_{x_k}$ vaguely satisfy the relation $\mathbb{R}$, then there is some compatible $f : \mathcal{P}(\mathbb{A}_{x_1}) \sqcup \cdots \sqcup \mathcal{P}(\mathbb{A}_{x_k}) \to S$, so
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- So $\hat{s}$ is a solution to $\mathbf{X}$!

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- The variables of this $\mathbf{X}$ correspond to the elements v of $\mathcal{V}(A)$, with variable domain $\mathbb{A}_v$ equal to the support of v.

- We impose a constraint $((v_1, \ldots, v_k), \mathbb{R})$ in $\mathbf{X}$ whenever $\mathbb{R} \leq_{sd} \mathbb{A}_{v_1} \times \cdots \times \mathbb{A}_{v_k}$ is vaguely satisfied by $v_1, \ldots, v_k$.

Back to robust satisfaction

Theorem (Z.)

If the semidefinite programming relaxation of an instance $\mathbf{X}$ *of* $\mathrm{CSP}(\mathbf{A})$ *has value* $1-\epsilon$*, then we can algorithmically find a vague solution to* $\mathbf{X}$ *which vaguely satisfies a* $1-f(\epsilon)$ *fraction of the constraints in polynomial time, where*

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- Once we have the (approx.) vague solution, we apply a vague rounding scheme to get an actual (approx.) solution.
- This is best possible: we can't robustly solve HORN-SAT with $f(\epsilon) = o(1/\log(1/\epsilon))$ unless the Unique Games Conjecture is false, by a result of Guruswami and Zhou.

Thank you for your attention.