Spirals

Zarathustra Brady

Clone-minimal algebras

- A *reduct* of $\mathbb{A}$ is an algebra with the same underlying set as $\mathbb{A}$ and basic operations a subset of the terms of $\mathbb{A}$. A reduct of $\mathbb{A}$ is *proper* if it is not term equivalent to $\mathbb{A}$, and *nontrivial* if at least one operation is not a projection.

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- Proposition

  Every nontrivial finite algebra $\mathbb{A}$ has a reduct which is clone-minimal. Any clone-minimal algebra $\mathbb{A}$ generates a variety in which all nontrivial members are clone-minimal.

Clone-minimal algebras which are Taylor

Theorem (Z.)

Suppose $\mathbb{A}$ is a finite algebra which is both clone-minimal and Taylor. Then one of the following is true:

1. *$\mathbb{A}$ is the idempotent reduct of a vector space over $\mathbb{F}_p$ for some prime p,*
2. *$\mathbb{A}$ is a minimal majority algebra, or*
3. *$\mathbb{A}$ is a minimal spiral.*

Spirals

- Definition
  An algebra $\mathbb{A} = (A, f)$ is a *spiral* if f is binary, idempotent, commutative, and for any $a, b \in \mathbb{A}$ either $\{a, b\}$ is a two element subalgebra of $\mathbb{A}$, or $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$ has a surjective map to the free semilattice on two generators.

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  - If $\mathbb{A}$ is a spiral of size at least three and $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}\{a, b\}$, then setting $S = \mathbb{A} \setminus \{a, b\}$ the definition implies that S binary-absorbs $\mathbb{A}$ and $f(a, b) \in S$.

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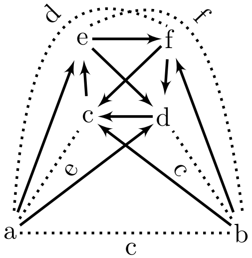
  - If $\mathbb{A}$ is a spiral of size at least three and $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}\{a, b\}$, then setting $S = \mathbb{A} \setminus \{a, b\}$ the definition implies that S binary-absorbs $\mathbb{A}$ and $f(a, b) \in S$.
  - Any 2-semilattice is a minimal spiral.

My first spiral



	a	b	c	d	e	f
a	a	c	e	d	e	d
b	c	b	c	c	f	f
c	e	c	c	c	e	c
d	d	c	c	d	d	d
e	e	f	e	d	e	f
f	d	f	c	d	f	f

Figure : A minimal spiral which is not a 2-semilattice.

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- *Step 0:* $\mathbb{A}$ is idempotent, since otherwise $\mathbb{A}$ has a nontrivial unary term φ, which generates a nontrivial non-Taylor clone.
- *Step 1:* Suppose there is some $\mathbb{B} \in HSP(\mathbb{A})$ which has a Mal'cev term m, that is, a term satisfying $m^{\mathbb{B}}(x, y, y) = m^{\mathbb{B}}(y, y, x) = x$ for all $x, y \in \mathbb{B}$.

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- Then $m(x, y, y) \approx m(y, y, x) \approx x$ in the variety generated by $\mathbb{A}$: if not, then $m(x, y, y)$ or $m(y, y, x)$ would generate a nontrivial proper reduct.

Proving the classification theorem: Mal'cev case

- Suppose that f, g are two n-ary terms of $\mathbb{A}$ with

$$f^{\mathbb{B}}(x_1, \ldots, x_n) = g^{\mathbb{B}}(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in \mathbb{B}$.

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- Then we must have

$$m(y, f(x_1, ..., x_n), g(x_1, ..., x_n)) \approx y$$

in the variety generated by $\mathbb{A}$, since otherwise the left hand side generates a nontrivial proper reduct.

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- Thus we have
$$g \approx m(f, f, g) \approx f,$$
so $\mathbb{A}$ and $\mathbb{B}$ generate the same variety. In particular, if $\mathbb{B}$ is the idempotent reduct of a vector space over $\mathbb{F}_p$, then so is $\mathbb{A}$.

Proving the classification theorem: bounded width case

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- Theorem (Larose, Valeriote, Zádori; Bulatov; Barto, Kozik)
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- Theorem (Jovanović, Marković, McKenzie, Moore)
  If $\mathbb{A}$ is a finite idempotent algebra of bounded width, then $\mathbb{A}$ has terms f_3, g satisfying the identities

$$f_3(x,y,y) \approx f_3(x,x,y) \approx f_3(x,y,x)$$
$$\approx g(x,x,y) \approx g(x,y,x) \approx g(y,x,x).$$

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- Theorem (Z.)

  If $\mathbb{A}$ *is a finite idempotent algebra of bounded width, then* $\mathbb{A}$ *has terms* f, g *satisfying the identities*

$$\begin{aligned} f(x,y) &\approx f(f(x,y), f(y,x)) \\ &\approx g(x,x,y) \approx g(x,y,x) \approx g(y,x,x). \end{aligned}$$

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  - Take terms f_3^1, g^1 from the previous theorem. Define f_3^i, g^i by

$$\begin{aligned} f_3^{i+1}(x,y,z) &= f_3^i(f_3(x,y,z), f_3(y,z,x), f_3(z,x,y)), \\ g^{i+1}(x,y,z) &= g^i(f_3(x,y,z), f_3(y,z,x), f_3(z,x,y)), \end{aligned}$$

    and choose $N \geq 1$ such that $f_3^N \approx f_3^{2N}$. Then take $g = g^N$.

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  we see that for any $a, b \in \mathbb{A}$, either $f(a,b) = f(b,a)$ or $\{f(a,b), f(b,a)\}$ is a majority subalgebra of $\mathbb{A}$.

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- Otherwise, f is nontrivial. If there was any majority algebra $\mathbb{B} \in HSP(\mathbb{A})$, then $f^{\mathbb{B}}$ would be a projection.

- Thus, if $\mathbb{A}$ is not a majority algebra, then there is no majority algebra $\mathbb{B} \in HSP(\mathbb{A})$, and so we must have

$$f(x,y) \approx f(y,x).$$

Proving the classification theorem: spiral case

- *Step 3:* Now we assume that $\mathbb{A} = (A, f)$ with f binary, idempotent, and commutative, such that $\mathbb{A}$ has bounded width.

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- By clone-minimality, if $(a, a) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$, then we must have $f(a, b) = f(b, a) = a$ and $\{a, b\}$ is a semilattice.

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- We want to show that $\mathbb{A}$ has a two-element semilattice subalgebra.

Proving there is a semilattice subalgebra

- Lemma
  Suppose that $\mathbb{A} = (A, f)$ *with* f *binary, idempotent, commutative, and suppose that* $\mathbb{A}$ *has no proper subalgebras. If* $(a, a) \notin \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$ *for all* $a \neq b \in \mathbb{A}$*, then* $\mathbb{A}$ *is affine.*

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  - Let $\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$. If $\mathbb{R}$ had any forks, then we'd get either $(a, a) \in \mathbb{R}$ or $(b, b) \in \mathbb{R}$, so $\mathbb{R}$ is the graph of an isomorphism $\iota_{a,b}$.

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  - Since $(f(a, b), f(a, b)) \in \mathbb{R}$, $\iota_{a,b}$ fixes $f(a, b)$.
  - $\mathrm{Aut}(\mathbb{A})$ is transitive, no nonidentity element of $\mathrm{Aut}(\mathbb{A})$ fixes more than one point, and $\forall a, b \in \mathbb{A}$ there is $\iota_{a,b} \in \mathrm{Aut}(\mathbb{A})$ of order two which swaps a, b and has one fixed point.

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  - So $\mathrm{Aut}(\mathbb{A})$ is a Frobenius group, and the Frobenius complement is an odd order abelian group.

Semilattice Iteration Lemma

- Lemma (Bulatov)

  Let t be a binary idempotent term of a finite algebra. Then there exists a nontrivially defined binary term $s \in \mathrm{Clo}(t)$ which satisfies the identities

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  - For any term t, let $t^1 = t$ and $t^{i+1}(x, y) = t(x, t^i(x, y))$. Set

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  - Now take $s(x, y) = u^\infty(x, y)$.

Theorem of the cube

- Suppose that s satisfies the identities

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Define a directed graph with an edge from a to b whenever $s(a, b) = b$. Note that there is an edge from a to b if and only if $\{a, b\}$ is closed under s, and s acts like the semilattice operation directed from a to b on $\{a, b\}$.

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- Theorem (Bulatov)

  If $R \subseteq A \times B \times C$ is closed under s, A, B, C are finite and strongly connected, and $\pi_{1,2}R = A \times B, \pi_{1,3}R = A \times C, \pi_{2,3}R = B \times C$, then $R = A \times B \times C$.

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- The proof is a generalization of the 2-semilattice case.

Back to classification theorem (spiral case)

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- Define a directed graph $\mathcal{G}_{\mathbb{A}}$ on A where edges correspond to two element semilattice subalgebras.
- For any a, b, either $s(a, b) = a$ or $(a, s(a, b)) \in \mathcal{G}$.

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- Since $f \in \mathrm{Clo}(s)$ and $x \rightarrow s(x, y)$, there is either a directed path from x to $f(x, y)$ or a directed path from y to $f(x, y)$. Since $f(x, y) \approx f(y, x)$, *both* directed paths exist.

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- So $\mathcal{G}_{\mathbb{A}}$ is connected. Moreover, for every algebra $\mathbb{B} \in HSP(\mathbb{A})$, $\mathcal{G}_{\mathbb{B}}$ has a unique maximal strongly connected component $S_{\mathbb{B}}$, and $S_{\mathbb{B}}$ is a binary absorbing subalgebra of $\mathbb{B}$.

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- Let $p(x, y)$ be in the maximal strongly connected component of the free algebra on two generators. Since $f \in \text{Clo}(p)$, $f(a, b)$ is in the maximal strongly connected component of $\text{Sg}\{a, b\}$ for any a, b.

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- Now assume $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}\{a, b\}$ with $|\mathbb{A}| > 2$, and let S be the maximal strongly connected component of $\mathcal{G}_{\mathbb{A}}$, so $\mathbb{A} = S \cup \{a, b\}$.

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  - We'll prove this using the Absorption Theorem.

  Theorem (Barto, Kozik)

  Suppose $\mathbb{A}, \mathbb{B}$ are finite algebras in a Taylor variety and $\mathbb{R}$ is a linked subdirect product of $\mathbb{A}$ and $\mathbb{B}$. Then either $\mathbb{R} = \mathbb{A} \times \mathbb{B}$ or one of $\mathbb{A}, \mathbb{B}$ has a proper absorbing subalgebra.

Proving the classification theorem: spiral case

- Now assume $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}\{a, b\}$ with $|\mathbb{A}| > 2$, and let S be the maximal strongly connected component of $\mathcal{G}_{\mathbb{A}}$, so $\mathbb{A} = S \cup \{a, b\}$.

- Lemma
  In this case, $S \cap \{a, b\} = \emptyset$, so $\mathbb{A}$ has a surjective map to the free semilattice on two generators.

  - We'll prove this using the Absorption Theorem.

  Theorem (Barto, Kozik)

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  - A strongly connected algebra has no proper absorbing subalgebras.

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- Let $\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$. If $\mathbb{R}$ is linked, then by the Absorption Theorem we have $\mathbb{R} = \mathbb{A} \times \mathbb{A}$, so $(b, b) \in \mathbb{R}$.

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- If $\mathbb{R}$ is not linked, $\mathbb{R}$ must be the graph of an isomorphism which swaps a and b. Now consider

$$\mathbb{B} = \mathrm{Sg}_{\mathbb{A}^3}\{(a, a, b), (a, b, a), (b, a, a)\}.$$

  Have $\pi_{i,j}\mathbb{B} = \mathbb{A} \times \mathbb{A}$ for all i, j, so $\mathbb{B} = \mathbb{A}^3$ by the theorem of the cube. If m witnesses the fact that $(b, b, b) \in \mathbb{B}$, then m restricts to a minority operation on $\{a, b\}$.

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- Let $\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\} \cap S^2$. Our assumption implies $\mathbb{R}$ must either be linked or the graph of an automorphism of S.
- If $\mathbb{R}$ linked, then by the Absorption Theorem have $(b,b) \in \mathbb{R}$.
- Otherwise, $\mathbb{R}$ is the graph of an automorphism $\iota : S \to S$. For any $x \in S$, have

$$(f(a,x), f(b,\iota(x))) \in \mathbb{R},$$
$$(f(\iota(b),x), f(b,\iota(x))) \in \mathbb{R},$$

so we must have $f(a,x) = f(\iota(b),x)$ for all $x \in S$. But then b and $\iota(b)$ generate S.

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- Proposition
  Every nontrivial reduct of a finite spiral is a bounded width algebra having no majority subalgebras. In particular, every nontrivial reduct of a finite spiral has a spiral term.

Thank you for your attention.