A few fewnomials

1. Prove that for $x \geq 0$, we have
$$x^7 + x^4 + x^3 + 1 \geq 2x^6 + 2x.$$

2. Prove that for $x \geq 0$, we have
$$x^{\sqrt{2}} + 2\sqrt{2} \geq 2^{\frac{3-\sqrt{2}}{2}} x + 2.$$

3. Prove that for all x we have
$$3x^2 e^x + 4e^2 \geq 8xe^x.$$

4. Prove that for $x \geq 0$, we have
$$x^{22} + x^{11} + x^9 + 1 \geq x^{21} + x^{15} + x^4 + x^2.$$

5. Prove that for $x > 0$, we have
$$x^{\sqrt{2}} + 2 + \frac{1}{x^{\sqrt{2}}} \geq 2x + \frac{2}{x}.$$

6. Prove that for $x \geq 0$, we have
$$x^9 + 281x^3 + 100 \geq 22x^6 + 360x^2.$$

7. For $x, y > 0$, define their *logarithmic mean* to be
$$\mathrm{LM}(x, y) = \frac{x - y}{\ln(x) - \ln(y)},$$
where $\ln(x)$ is the natural logarithm of x. Prove that
$$\frac{x + y}{2} \geq \mathrm{LM}(x, y) \geq \sqrt{xy}.$$

8. Prove that for any $x > 0$ we have
$$x^{66} + x^{29} + x^{26} + \frac{1}{x^{26}} + \frac{1}{x^{29}} + \frac{1}{x^{66}} \geq x^{62} + x^{45} + x^2 + \frac{1}{x^2} + \frac{1}{x^{45}} + \frac{1}{x^{62}}.$$

9. Prove that if f is differentiable and f' is convex, then we have

$$f(3) + 3f(1) \geq 3f(2) + f(0), \text{ and}$$
$$f(6) + f(2) + f(1) \geq f(5) + f(4) + f(0).$$

10. (Vasc) Prove that if f is differentiable and f' is convex, then for any $x \geq y \geq z$ we have

$$f(2x + y) + f(2y + z) + f(2z + x) \geq f(2x + z) + f(2z + y) + f(2y + x).$$

11. Popoviciu defines the divided differences of a polynomial inductively, as follows:

$$[a; f] = f(a),$$
$$[a, b; f] = \frac{f(b) - f(a)}{b - a},$$
$$[a_0, ..., a_n; f] = \frac{[a_1, ..., a_n; f] - [a_0, ..., a_{n-1}; f]}{a_n - a_0}.$$

Prove, by induction on n, that if the nth derivative of f exists and is nonnegative, that for any $a_0, ..., a_n$ we have

$$[a_0, ..., a_n; f] \geq 0.$$

12. Show that $[a_0, ..., a_n; f]$ is a symmetric function of $a_0, ..., a_n$.

13. Let n be an integer which is at least 3. Suppose f is a function such that for every $a_0, ..., a_n$ we have $[a_0, ..., a_n; f] \geq 0$. Show that f is differentiable and that for any $b_0, ..., b_{n-1}$ we have

$$[b_0, ..., b_{n-1}; f'] \geq 0.$$

14. Suppose that f is a function such that for every integer n and every $a_0, ..., a_n$ we have $[a_0, ..., a_n; f] \geq 0$. Prove that

$$f(2) + 4f(0) \geq 4f(1).$$

15. Prove that if $a_1, ..., a_9$ are real numbers satisfying $\sum_{i=1}^{9} a_i = \sum_{i=1}^{9} a_i^3 = 0$ and $\sum_{i=1}^{9} a_i^2 = 8$, then for any $x > 0$ we have

$$x^2 + 7 + \frac{1}{x^2} \geq \sum_{i=1}^{9} x^{a_i} \geq 4x + 1 + \frac{4}{x}.$$

16. Prove that for any $x, y, z > 0$ we have

$$\sum_{sym} \frac{x^2}{y^2} + \sum_{sym} \frac{x^{\sqrt{2}}}{y^{\sqrt{2}}} \geq \sum_{sym} \frac{x^2}{yz} + \sum_{sym} \frac{xy}{z^2}.$$