Simplifying clones with partial semilattice operations

Zarathustra Brady

Unary iteration

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Proposition

There is some m dividing $\mathrm{lcm}\{1, 2, ..., |A|\}$ such that

$$f^{\circ m}(x) \approx f^{\circ km}(x)$$

for all $k \geq 1$.

Unary iteration, continued

Definition

For $f : A \to A$, $|A| < \infty$, define $f^{\circ\infty}$ by

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- For any f, $f^{\circ\infty}$ satisfies

$$f^{\circ\infty}(f^{\circ\infty}(x)) \approx f^{\circ\infty}(x).$$

- If $e : A \to A$ satisfies

$$e(e(x)) \approx e(x),$$

we say that e is *compositionally idempotent*.

Nice behavior of unary iteration

- The map $f \mapsto f^{\circ\infty}$ is compatible with homomorphisms:

$$\begin{array}{ccc} (A, f) & \xrightarrow{\varphi} & (B, g) \\ \downarrow & & \downarrow \\ (A, f^{\circ\infty}) & \xrightarrow{\varphi} & (B, g^{\circ\infty}) \end{array}$$

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- Also compatible with finite products.

- As a bonus, $f^{\circ\infty}$ can be computed from f in $O(|A|)$ steps.

Using compositionally idempotent unary operations

- If $e : A \to A$ is compositionally idempotent and $f : A^n \to A$, set
$$f_e(x_1, \ldots, x_n) := e(f(e(x_1), \ldots, e(x_n))).$$

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- Have $f_e \in \mathrm{Clo}(e, f)$ and
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- If
$$f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m),$$
then
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then
$$f_e(x_1, ..., x_n) \approx g_e(y_1, ..., y_m).$$
- The map $f \mapsto f_e$ preserves identities of height at most one, and shrinks the domain.

Reduction to cores

- If we are studying identities of height one, we can replace $\mathbb{A}$ by

$$\mathbb{A}_e := (e(A), \{f_e\}_{f \in \mathrm{Clo}(\mathbb{A})})$$

for any $e \in \mathrm{Clo}_1(\mathbb{A})$ which is compositionally idempotent.

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- Eventually, we reduce to the case where

$$f^{\circ\infty}(x) \approx x$$

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- In this case, $\mathrm{Clo}_1(\mathbb{A})$ must be a group!

Reduction to idempotent algebras

- If $\text{Clo}_1(\mathbb{A})$ is a group, then $f \in \text{Clo}(\mathbb{A})$ can be decomposed:

$$f(x_1, \dots, x_n) \approx f_{un}(f_{id}(x_1, \dots, x_n)),$$

where

$$f_{un}(x) := f(x, \dots, x)$$

is unary and invertible, and

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- Starting from $t : A^2 \to A$, we will construct $s \in \mathrm{Clo}_2(t)$ satisfying

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- We call such an s a *partial semilattice operation*.
- We will use partial semilattice operations s to simplify our clones (while preserving some height one identities).
- When no further simplifications are possible, binary absorption will have nice properties.

Binary iteration: the first step

- For $t : A^2 \to A$, define $t^{\circ_2 n}$ by

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- We automatically have

$$t^{\circ_2 \infty}(x, t^{\circ_2 \infty}(x, y)) \approx t^{\circ_2 \infty}(x, y).$$

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- Then
$$f(x, u(x, y)) \approx u(x, y),$$
so
$$\begin{aligned} u(u(x, y), x) &\approx f(u(x, y), f(x, u(x, y))) \\ &\approx f(u(x, y), u(x, y)) \\ &\approx u(x, y). \end{aligned}$$

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- Taking the limit, we get

$$s(s(x,y),x) \approx s(x,y) \approx s(x,s(x,y)).$$

Binary iteration: putting it all together

- Our full construction is given by

$$\begin{aligned} f(x,y) &:= t^{\circ_2 \infty}(x,y), \\ u(x,y) &:= f(x, f(y,x)), \\ s(x,y) &:= u^{\circ_2 \infty}(x,y). \end{aligned}$$

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- More compactly:

$$s := t^{\circ_2 \infty}(\pi_1, t^{\circ_2 \infty}(\pi_2, \pi_1))^{\circ_2 \infty}.$$

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- Also compatible with finite products.
- As a bonus, s_i can be computed from t_i in time $O(|A_i|^2)$.

Compatibility with binary absorption

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$$t(C, B), t(B, C) \subseteq C$$

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- In particular,

$$t(a, b) = t(b, a) = b \implies s(a, b) = s(b, a) = b.$$

Meaning of the partial semilattice identities

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$$(\{x, s(x, y)\}, s)$$

  being a semilattice with absorbing element $s(x, y)$.

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- We have

$$\begin{aligned} a \rightarrow_s b \quad &\iff \quad s(a, b) = b \\ &\iff \quad \exists c \text{ s.t. } s(a, c) = b. \end{aligned}$$

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- We want to find a reduct of $\mathbb{A}$ which satisfies the properties above, which preserves the height one identities satisfied by $\mathbb{A}$.

- It isn't possible to preserve all height one identities: they must be compatible with semilattices.

Two-variable height-one identities

- For every $n \geq 2$, define $s_n : A^n \to A$ by

$$s_n(x_1, ..., x_n) := s(s_{n-1}(x_1, ..., x_{n-1}), s(x_1, x_n)).$$

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- For $f : A^n \to A$, define f_s by
$$f_s(x_1, \ldots, x_n) := f(s(x_1, \ldots, x_n), s(x_2, \ldots, x_n, x_1), \ldots, s(x_n, x_1, \ldots, x_{n-1})).$$

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- If
$$f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$$
and $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_m\} = \{x, y\}$, then
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- If $a \to_s b$, then $\{a, b\}$ is a subalgebra of $\mathbb{A}_s$, term equivalent to $(\{a, b\}, s)$.
- Every system of two-variable height-one identities with both variables occuring on each side which is satisfied in $\mathbb{A}$ is also satisfied in $\mathbb{A}_s$.

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- Every system of two-variable height-one identities with both variables occuring on each side which is satisfied in $\mathbb{A}$ is also satisfied in $\mathbb{A}_s$.
- In particular:
  - If $\mathbb{A}$ is Taylor, then $\mathbb{A}_s$ is also Taylor.
  - If $\mathbb{A}$ has bounded width, then $\mathbb{A}_s$ also has bounded width.

Symmetric operations

- An operation $f : A^n \to A$ is *symmetric* if

$$f(x_1, ..., x_n) = f(x_{\sigma(1)}, ..., x_{\sigma(n)}).$$

for all permutations $\sigma \in S_n$.

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- $\mathbb{A}$ has symmetric operations of every arity iff the Linear Programming relaxation solves CSP($\mathbb{A}$).
- If f_n are a system of symmetric operations for each arity n, write
$$f_n^s(x_1, \ldots, x_n) := f_{n!}(s_n(x_{\sigma_1(1)}, \ldots, x_{\sigma_1(n)}), \ldots, s_n(x_{\sigma_{n!}(1)}, \ldots, x_{\sigma_{n!}(n)})).$$

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for all permutations $\sigma \in S_n$.
- $\mathbb{A}$ has symmetric operations of every arity iff the Linear Programming relaxation solves CSP($\mathbb{A}$).
- If f_n are a system of symmetric operations for each arity n, write
$$f_n^s(x_1, ..., x_n) := f_{n!}(s_n(x_{\sigma_1(1)}, ..., x_{\sigma_1(n)}), ..., s_n(x_{\sigma_{n!}(1)}, ..., x_{\sigma_{n!}(n)})).$$
- Each f_n^s is symmetric, and if $a \to_s b$ then f_n^s acts like s_n on $\{a, b\}$.

Totally symmetric operations

- An operation $f : A^n \to A$ is *totally symmetric* if

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- $\mathbb{A}$ has totally symmetric operations of every arity iff Arc Consistency solves CSP($\mathbb{A}$).

Proposition

If $\mathbb{A}$ has totally symmetric operations f_n of every arity n, then there are totally symmetric operations $f_n^s \in \mathrm{Clo}(\mathbb{A})$ such that if $a \to_s b$ then f_n^s acts like s_n on $\{a, b\}$.

Analogue of idempotence

- These constructions involved preprocessing the inputs to functions $f \in \mathrm{Clo}(\mathbb{A})$ by applying the operations s_n.

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- I say that an algebra $\mathbb{A}$ has been *prepared* if

$$\begin{bmatrix} b \\ b \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^2} \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \right\}$$

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- If $\mathbb{A}$ is prepared, then we write $a \to b$ if the above holds.

Binary absorption and strong absorption

- Write $\mathbb{B} \lhd_{bin} \mathbb{A}$ ($\mathbb{B}$ *binary absorbs* $\mathbb{A}$) if there is a binary term t such that $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to t.

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$$\mathbb{B} \lhd_{bin} \mathbb{C} \implies \mathbb{B} \lhd_{str} \mathbb{C}.$$
- The previous constructions can be used to reduce to the case where $\mathbb{A}$ is strongly prepared.

Transitivity of binary absorption?

- Suppose that $\mathbb{B} \lhd_{bin} \mathbb{A}$ and $\mathbb{C} \lhd_{bin} \mathbb{B}$. Does it follow that $\mathbb{C} \lhd_{bin} \mathbb{A}$?

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- In general, no:

$$\mathbb{A} = (\{0,1\}^2, \wedge, \vee),$$
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- Transitivity also fails for strong absorption:

$$\{c\} \lhd_{str} \{b,c\} \lhd_{str} \{a,b,c\}$$

in the idempotent commutative groupoid with $ab = ac = b$ and $bc = c$.

Useful lemma about absorption

Lemma

If $\mathbb{A}$ is prepared, and if $\mathbb{B} \lhd \mathbb{A}$, then for any partial semilattice operation $s \in \mathrm{Clo}_2(\mathbb{A})$ we have

$$s(\mathbb{B}, \mathbb{A}) \subseteq \mathbb{B}.$$

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If $\mathbb{A}$ is prepared, and if $\mathbb{B} \triangleleft \mathbb{A}$, then for any partial semilattice operation $s \in \mathrm{Clo}_2(\mathbb{A})$ we have

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Proof.

If $b \in \mathbb{B}$ and $s(b, a) \notin \mathbb{B}$, then $\{b, s(b, a)\}$ is a subalgebra of $\mathbb{A}$ which is not absorbed by $\{b\} = \mathbb{B} \cap \{b, s(b, a)\}$. □

Preparation fixes transitivity

Proposition

If $\mathbb{A}$ *is prepared, and if* $\mathbb{C} \lhd_{bin} \mathbb{B} \lhd_{bin} \mathbb{A}$*, then* $\mathbb{C} \lhd_{bin} \mathbb{A}$*.*

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Proof.

Choose a partial semilattice term s such that

$$s(\mathbb{B}, \mathbb{C}), s(\mathbb{C}, \mathbb{B}) \subseteq \mathbb{C},$$

and any t witnessing $\mathbb{B} \lhd_{bin} \mathbb{A}$. Define $u \in \mathrm{Clo}(s, t)$ by

$$u(x, y) := s(s(t(x, y), y), s(t(y, x), x)).$$

Then u witnesses $\mathbb{C} \lhd_{bin} \mathbb{A}$. □

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If $\mathbb{A}$ *is prepared, and if* $\mathbb{B}_1 \lhd \mathbb{A}, \mathbb{B}_2 \lhd_{bin} \mathbb{A}$*, then* $\mathbb{B}_1 \cap \mathbb{B}_2 \neq \emptyset$*.*

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Proof.

Choose a partial semilattice term s such that

$$s(\mathbb{B}_2, \mathbb{A}), s(\mathbb{A}, \mathbb{B}_2) \subseteq \mathbb{B}_2,$$

and any $b_1 \in \mathbb{B}_1$, $b_2 \in \mathbb{B}_2$. Then

$$s(b_1, b_2) \in \mathbb{B}_1 \cap \mathbb{B}_2.$$

□

Nice criterion for binary absorption

Proposition

If $\mathbb{A}$ *is prepared, and if* $s(\mathbb{B}, \mathbb{A}) \subseteq \mathbb{B}$ *for a partial semilattice operation* s*, then TFAE:*

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If $\mathbb{A}$ is prepared, and if $s(\mathbb{B}, \mathbb{A}) \subseteq \mathbb{B}$ for a partial semilattice operation s, then TFAE:

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- *for all $a \in \mathbb{A} \setminus \mathbb{B}$ and $b \in \mathbb{B}$, there is some $b' \in \mathrm{Sg}_{\mathbb{A}}\{a, b\}$ such that*

$$a \to b' \in \mathbb{B}.$$

Thank you for your attention.