Visible obstructions, parallelograms, and increasing subsequences.

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1 Introduction

We are concerned with the following question:

Suppose you are standing on a square within an n by n square grid, and that some of the grid squares are entirely filled by opaque obstructions. Call an obstruction visible if you can see any part of it from any point in the square you are standing in. Assuming the obstructions are placed optimally, what is the largest possible number of visible obstructions?

Specifically, we are interested in finding the order of growth (up to constants) of the largest possible number of visible obstructions. More precisely, if we let $f(n)$ be the number of visible obstructions, we would like to find a simple function $p(n)$ (such as n, n^2, or $n\log(n)$) and show that there are constants c_1, c_2 such that

$$c_1p(n) \le f(n) \le c_2p(n)$$

for all sufficiently large n. As we will see, the order of growth (up to constants) for $f(n)$ turns out to be $p(n) = n^{3/2}$.

Since we only care about the number of visible obstructions up to a constant factor, we can simplify the problem in various ways. For instance, we may assume that we are standing in a corner of the grid, and multiply the maximum number of obstructions visible from the corner of the grid by a factor of four to get an upper bound on the number of obstructions visible from a square in the interior of the grid. We may also assume that only squares with coordinates a multiple of some fixed constant m may be covered by obstructions, by constructing an n/m by n/m grid and considering a square of the new grid to be obstructed if the corresponding m by m subgrid of our original grid contained a visible obstruction. By taking $m = 2$ and mentally rotating the grid by 45 degrees, we can assume that obstructions are diamonds connecting the midpoints of the sides of our grid squares. The advantage of this point of view is that when we restrict ourselves to the first octant of the grid (from the point of view of someone standing in the grid), diamonds may be replaced with vertical lines through the midpoints of the top and bottom segments of the grid squares.

From now on, we will therefore work on the simplified problem of counting obstructing vertical line segments which have integer coordinates less than or equal to n, are in the first octant of the plane, and are visible from the vertical segment connecting $(0, 0)$ to $(0, 1)$.

2 The parallelogram trick

The first step is to find a good way to check if a vertical segment v is visible from the vertical segment v_0 at the origin. To do this, we draw the parallelogram connecting the vertical segment v to the vertical segment at the origin, and note that if we can see the segment v from a point on v_0, then we can draw a "sight line" within this parallelogram from a point on v_0 to a point on v that doesn't intersect any other obstructing line segments. The key point is that the only obstructing vertical segments we have to consider are those that intersect this parallelogram, and since this parallelogram has height one, every vertical segment intersecting it either intersects the top of the parallelogram, or intersects the bottom of the parallelogram.

Thus we can divide the obstructions intersecting this parallelogram into "upper" and "lower" obstructions. Now define the "upper convex hull" to be the convex hull of the upper obstructions, along with the top point of v_0 and the top point of v, and define the "lower convex hull" similarly. Then any potential sight line from v_0 to v must separate the upper and lower convex hulls, and conversely if the upper and lower convex hulls do not intersect then we can find a line separating them, which must pass through the boundary of the parallelogram at v_0 and again at v. Thus we can see that the segment v is visible from the segment v_0 if and only if the upper and lower convex hulls do not intersect.

3 Construction for the lower bound

To construct a placement of obstructions, we use the idea of dividing the first octant of the plane into parallelograms that interact as little as possible. Specifically, we assume that n is a multiple of 4, say $n = 4p$, and consider the parallelograms connecting the segment v_0 to the vertical segment connecting $(4p, 4q)$ to $(4p, 4q + 1)$. It's clear then that when the x-coordinate is at least $2p$, the distance between two consecutive parallelograms is at lest 1, so past this point vertical line segments intersect at most one parallelogram. Furthermore, our construction will only involve "lower" obstructions in each parallelogram.

Fix a parallelogram of slope q/p, and for simplicity assume p is prime (by Bertrand's postulate, we can always find a prime between p and $2p$, so we are only losing a constant factor of accuracy in doing this). Let v_j be the line segment with x-coordinate j intersecting the bottom of the parallelogram, then the vertical distance from the top point of the segment v_j to the bottom of the parallelogram is $1 - \{\frac{qj}{p}\}$. If we only consider sight lines parallel to the bottom side of this parallelogram, then a sequence of line segments v_j will all be visible as long as the corresponding sequence of heights $1 - \{\frac{qj}{p}\}$ is increasing with j, or equivalently the sequence $\{\frac{qj}{p}\} = 1 - \{\frac{(p-q)j}{p}\}$ is decreasing with j.

If we consider the parallelograms with slopes q/p and $(p - q)/p$ together, we see that a lower bound for the number of visible obstructions that we can fit in these two parallelograms is the sum of the length of the longest increasing subsequence of $1 - \{\frac{qj}{p}\}$ and the length of the longest decreasing subsequence of $1-\{\frac{qj}{p}\}$, j running from $2p+1$ to $3p-1$. By the Erdös-Szekeres Theorem [1], the length of the longest monotone subsequence of $1 - \{\frac{qj}{p}\}$ is at least $\sqrt{p-1}$, and the number of such pairs of parallelograms is $\frac{p+1}{2}$, so we can fit at least $p^{3/2}/2$ visible obstructions into this octant.

4 Upper bound

To get an upper bound on the number of visible obstructions, we note that we can cover the first octant with $n+1$ parallelograms, going from the vertical segment at the origin to the vertical segments with x-coordinate n. Then any visible obstruction contains a point that is visible and this point will be contained in one of those $n+1$ parallelograms, and this visible obstruction will either occur as a visible upper obstruction or a visible lower obstruction in this parallelogram. Thus we just have to bound the maximum number of visible upper and lower obstructions in each parallelogram.

For simplicity we assume that n is a prime p, and will bound the number of obstructions that occur as visible lower obstructions in one of these parallelograms (the analysis for upper obstructions is similar). Call a lower obstruction to be "visible as a lower obstruction" in a parallelogram if it is not blocked by any other lower obstructions contained in that parallelogram, and note that a lower obstruction is visible as a lower obstruction if and only if its top point is not blocked by other lower obstructions in this parallelogram. If an obstruction is a lower obstruction in more than one parallelogram, then if it is visible as a lower obstruction in the parallelogram with greater slope it must also be visible as a lower obstruction in the parallelogram with smaller slope. Call a lower obstruction a primitive lower obstruction of a parallelogram if it is not a lower obstruction in any parallelogram of smaller slope. By the above discussion, we only have to bound the number of primitive lower obstructions in each parallelogram.

Consider a parallelogram of slope q/p. Let the lower obstruction with x-coordinate j be v_j. Then the vertical distance from the top point of this lower obstruction to the bottom of this parallelogram is again $1-\{\frac{qj}{p}\}$, and a sequence of lower obstructions v_{j_k} is visible as a sequence of lower obstructions if and only if the sequence of slopes from the top point of v_0 to the top points of the v_{j_k}s is increasing, which occurs if and only if the sequence $\{\frac{qj_k}{p}\}/j_k$ is decreasing.

Thus, we just need to get a bound on the length of a decreasing subsequence of $\{\frac{qj}{p}\}/j$ for primitive lower obstructions v_j. The idea now is to find a jump length $l > 0$ such that

$$\{\frac{qj}{p}\}/j < \{\frac{q(j+l)}{p}\}/(j+l).$$

If we can find such an l, then we can split up the sequence $\{\frac{qj}{p}\}/j$ into a collection of l increasing subsequences, so the length of any decreasing subsequence is at most l. Note that if v_j is a *primitive* lower obstruction, then we have the inequality $\{\frac{qj}{p}\}/j < 1/p$, so if l satisfies the inequality $\{\frac{ql}{p}\}/l \geq 1/p$ then

$$\{\frac{qj}{p}\}/j < \left(\{\frac{qj}{p}\}+\{\frac{ql}{p}\}\right)/(j+l),$$

and the expression on the right is equal to $\{\frac{q(j+l)}{p}\}/(j+l)$ unless $\{\frac{qj}{p}\}+\{\frac{ql}{p}\} \geq 1$, which occurs for only $(ql \mod p)$ different "bad values" of j. We can similarly see that if v_j is not a primitive lower obstruction, then v_{j+l} is also not a primitive lower obstruction unless j is one of the $(ql \mod p)$ bad values. Thus, the sequence $\{\frac{qj}{p}\}/j$ for j such that v_j is a primitive lower obstruction splits into at most $l+(ql \mod p)$ increasing subsequences, so the length of the longest decreasing subsequence is at most $l+(ql \mod p)$.

All that remains is to show that for each q, we can find l such that $\{\frac{ql}{p}\}/l \geq 1/p$ and $l+(ql \mod p)$ is small. Let $m = (ql \mod p)$. Then we have $m/l \equiv q \mod p$, and $\{\frac{ql}{p}\}/l = \frac{m}{pl} \geq 1/p$

if and only if $m \geq l$. By a standard Pigeonhole argument on the first $\lceil\sqrt{p}\rceil$ multiples of q, we can always find i, j with $|i|, |j| \leq \lceil\sqrt{p}\rceil$ such that $i/j \equiv q \mod p$. We would like to convert the representation i/j into a representation m/l with $m \geq l > 0$, and $m + l$ small. If we don't already have $i \geq j > 0$, then we split into two cases:

First, suppose that i, j have opposite signs, say i is positive and j is negative. Then $\frac{i}{i-j} \equiv \frac{q}{q-1} \mod p$, and we have $2\lceil\sqrt{p}\rceil > i-j > i > 0$, so by the next case we can find m', l' such that $\frac{m'}{l'} \equiv \frac{i}{i-j} \mod p$ and $m' > l' > 0$, and then $\frac{m'}{m'-l'} \equiv q \mod p$, so we take $m = m', l = m' - l'$, and we have $m + l \leq 2m'$.

Second, suppose that $2\lceil\sqrt{p}\rceil > j > i > 0$, and that i, j are relatively prime. Let a be five times the multiplicative inverse of i mod j, so $\frac{ai-5}{j}$ is an integer. Let $k = \lceil\frac{ap}{j}\rceil$. Then $ap < jk < ap + j$, and $\frac{ai-5}{j}p + \frac{5p}{j} < ik < \frac{ai-5}{j}p + \frac{5p}{j} + i$, so

$$\frac{5p}{j} + i > (ik \mod p) > \frac{5p}{j} > j > (jk \mod p) > 0,$$

and we can take $m = (ik \mod p), l = (jk \mod p)$.

Now when we add up $m + l$ over all possible values of q, what we get is at most the sum over $\lceil\sqrt{p}\rceil \geq i \geq j > 0$ of $i + j$, plus the sum over $2\lceil\sqrt{p}\rceil > j > i > 0$ of $\frac{5p}{j} + i + j$ (for the second case above) and $2\frac{5p}{j} + 2i$ (for the first case above). Since each $i + j$ or $2i$ is at most $4\lceil\sqrt{p}\rceil$, and since the number of times each $\frac{5p}{j}$ appears in that sum is at most $3j$ ($1 + 2$ times for each choice of i less than j), we get that the sum is at most

$$p \times 4\lceil\sqrt{p}\rceil + \sum_{j=1}^{2\lceil\sqrt{p}\rceil} 3j \times \frac{5p}{j} = 34p\lceil\sqrt{p}\rceil,$$

so the total number of visible lower obstructions in the first octant is at most $34p\lceil\sqrt{p}\rceil$.

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References

[1] P. Erdös and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.