Ideal multigrades with trigonometric coefficients

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1 The problem

An (n,k) multigrade is defined as a pair of distinct sets of integers

$$(a_1, ..., a_n; b_1, ..., b_n)$$

such that

$$\sum_{i=1}^{n} a_i^j = \sum_{i=1}^{n} b_i^j$$

for all positive integers $j \le k$. A common notation for such sets is

$$a_1, ..., a_n \stackrel{k}{=} b_1, ..., b_n.$$

A solution is said to be ideal when $n = k + 1$, since it is easy to see that when $n = k$ the only solutions are trivial. The Tarry-Escott-Prouhet Problem is to prove (or disprove) that there is an ideal multigrade of degree k for every k.

2 Approach

Let X be the variety of ideal multigrades of degree k (in other words, the multigrades for which n, the number of variables in each set, is $k+1$). We identify a hyperplane of trivial multigrades with $\mathbb{P}^k$. Let C be any curve contained in that hyperplane. Then we have the inclusions

$$C \subset \mathbb{P}^k \subset X \subset \mathbb{P}^{2n-1},$$

and if we let R be the ring $\mathbb{Q}[x_0, ..., x_k, y_0, ..., x_k]$, I be the ideal of C in R, H the ideal of the hyperplane of trivial solutions, and J be the ideal of X, we have the reverse inclusions

$$R \supset I \supset H \supset J.$$

We want to study the deformations of C. According to [1], the space of first order deformations can be identified with the global sections of the normal sheaf to C. For the deformations that remain in $\mathbb{P}^k$, the normal sheaf is $\mathcal{N}_{C/\mathbb{P}^k} = \mathrm{Hom}(I/H, R/I)\tilde{}$, while for deformations that remain in X the normal sheaf is $\mathcal{N}_{C/X} = \mathrm{Hom}(I/J, R/I)\tilde{}$. Thus, if $\dim H^0(\mathcal{N}_{C/X}) > \dim H^0(\mathcal{N}_{C/\mathbb{P}^k})$, then the curve C has a nontrivial first order deformation. Hopefully, this first order deformation could then be extended to a full deformation, finally giving us a nontrivial rational curve contained in X.

3 Previous Results

If we restrict to the case where C is a rational curve defined by a homogenous degree d map $(s,t) \to (f_0(s,t), ..., f_k(s,t); f_0(s,t), ..., f_k(s,t))$ (i.e., $x_i = y_i = f_i(s,t)$), then as long as no two f_i are identical C has a nontrivial first order deformation within the variety X if and only if

$$\deg \gcd_i \left(\prod_{a<b, a,b \neq i} (f_a(s,t) - f_b(s,t)) \right) \geq d\frac{n(n-3)}{2}.$$

This is the result of our previous paper. It turns out that it is more useful to subtract both sides from $d\frac{(n-1)(n-2)}{2}$, so we get

$$\deg \mathrm{lcm}_i \left(\prod_{i \neq j} (f_i(s,t) - f_j(s,t) \right) \leq dn.$$

We can interpret this result geometrically as follows. Get a piece of $\mathcal{O}_{\mathbb{P}^1}(d)$ graph paper, and draw graphs of $f_0, ..., f_k$ on it. Mark every x-coordinate at which two functions f_i, f_j are equal, and then count them "with multiplicity" and compare this sum with nd. If it is smaller than or equal to nd, then the curve C has nontrivial first order deformations.

To count "with multiplicity," we can use the following rules of thumb:

- If there is more than one point over a single x coordinate where two graphs come together, we take the maximum of the multiplicities over all such points to get the multiplicity of this x coordinate.
- If all of the curves intersect transversally, then the multiplicity at a point with several curves passing through it is the number of curves passing through it minus one.

For more complicated situations, we can refer back to the algebraic definition above.

This paper will be concerned with the case $d = 1$, in which case the rules of thumb given above are enough to properly calculate the multiplicity.

4 Linear curves with nontrivial deformations

Since we are looking for rational solutions to the multigrade problem, we restrict to the case $f_i \in \mathbb{R}[s, t]$, and we can assume without loss of generality that no intersections occur at infinity. Now we can rephrase the problem in terms of counting x coordinates of intersections of nonparallel, nonvertical lines in the real plane.

I claim that if any three lines pass through a point, then all lines must pass through that point. To see this, assume the contrary, and perform a projective transformation on the x axis to make the x coordinate of this point occur between two other x-coordinates where intersections occur. Taking the two closest such x coordinates on either side of this x coordinate, we see that the middle line can not intersect any other line at either of these two x coordinates. Now if we count intersections with the middle line with multiplicity, we get $n-1$, and there are at least two additional x coordinates not involving the middle line, contradiction.

Thus we have two cases: the case where all lines go through a single point, and the case where all intersections have multiplicty one. If all lines go through a single point, we get the curve C given by $(s, t) \to (s + c_0 t, s + c_1 t, ..., s + c_k t; s + c_0 t, ..., s + c_k t)$ described in the previous paper.

Now we turn to the case where all intersections have multiplicity one. Call an x coordinate special if two lines intersect above it. Call a point on a line above this x coordinate full if it is the intersection of two lines, and call it empty otherwise. We see that there must be at least one empty point by looking at a minimal triangle formed by our lines, so there are necessarily more than $n - 1$ special x coordinates. Thus there are n special x coordinates, and there must be exactly n empty points.

Between two consecutive special x coordinates, all of the lines have a fixed ordering, so there is a line with a smallest height, and its height is strictly smallest at one of the two special x coordinates. Thus, the intersection of this line with that x coordinate gives us an empty point below all of the full points with this x coordinate. Similarly, for any two adjacent special x coordinates, at least one of them has an empty point above all of its full points. Since there are only n empty points, they must all be either above or below all the full points with the same x coordinate. This information together with the parity of n gives us the full incidence structure of our lines, which in turn gives us the equations of our lines up to linear equivalence.

Instead of solving for the equations of these lines, we can construct them as follows: take a line in 3 dimensional space that doesn't intersect the x axis, and rotate it around the x axis by the angles $\frac{2\pi i}{n}$ for $i = 0, ..., k$. Then project all of these lines onto the xy plane. This gives us the explicit equations

$$f_i(s,t) = \cos\left(\frac{2\pi i}{n}\right) s + \frac{\sin\left(\frac{2\pi i}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)} t.$$

These equations have coefficients in $\mathbb{Q}[\cos\left(\frac{2\pi}{n}\right)]$, which has degree $\phi(n)/2$ over the rationals. $n = 6$ is the last n for which this is defined over the rationals, and for $n = 6$ we can rewrite our six lines as $s, t, s+t, -s, -t, -s-t$ by a linear change of variables. Coincidentally, it is known [reference?] that we have

$$\pm s, \pm t, \pm(s+t) \stackrel{5}{=} \pm u, \pm v, \pm(u+v)$$

$$\text{if and only if}$$

$$s^2 + t^2 + st = u^2 + v^2 + uv.$$

Similarly, for n = 8 we get that

$$\pm a \pm b(\sqrt{2}+1), \pm a(\sqrt{2}+1) \pm b \stackrel{7}{=} \pm c \pm d(\sqrt{2}+1), \pm c(\sqrt{2}+1) \pm d$$

$$\text{if and only if}$$

$$a^2 + b^2 = c^2 + d^2.$$

In general, we have

Theorem 1.

$$f_0(s,t), ..., f_k(s,t) \stackrel{k}{=} f_0(u,v), ..., f_k(u,v)$$

$$\textit{if and only if}$$

$$f_0(s,t), ..., f_k(s,t) \stackrel{2}{=} f_0(u,v), ..., f_k(u,v)$$

Proof. Set $\zeta = e^{\frac{2\pi i}{n}}$. After a complex linear change of variables, we can write $f_i = \zeta^i s + \zeta^{-i} t$. Now for $j = 1, ..., k$, we have

$$\sum_i f_i^j = \sum_i \zeta^{ij} s^j + \binom{j}{1} \zeta^{i(j-2)} s^{j-1} t + \cdots + \zeta^{-ij} t^j,$$

which is 0 if j is odd and $n\binom{j}{j/2} s^{j/2} t^{j/2}$ if j is even. Thus $\sum_i f_i^j$ is completely determined by $st = (\sum_i f_i^2)/2n$ for $j \leq k$. □

References

[1] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.