Searching for Multigrades

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October 25, 2009

Abstract

An (n, k) multigrade is defined to be a pair of sets of n numbers that have equal sums, sums of squares, and so on up to kth powers. The Prouhet-Tarry-Escott problem is to find integer multigrades with $n = k + 1$ (called ideal multigrades). For $k \leq 7$, parametric solutions have been found to the multigrade problem. We attempt to find more parametric solutions by finding curves contained in the set of trivial solutions (i.e., both sets are the same), and deforming them out and into the nontrivial solutions.

1 The problem

An (n, k) multigrade is defined as a pair of distinct sets of integers

$$(a_1, ..., a_n; b_1, ..., b_n)$$

such that

$$\sum_{i=1}^{n} a_i^j = \sum_{i=1}^{n} b_i^j$$

for all positive integers $j \leq k$.

A solution is said to be ideal when $n = k + 1$, since it is easy to see that when $n = k$ the only solutions are trivial. The Tarry-Escott-Prouhet Problem is to prove (or disprove) that there is an ideal multigrade of degree k for every k.

Multigrades arise in a variety of situations: for instance, a polynomial of the form

$$\sum_{i=1}^{n} x^{a_i} - \sum_{i=1}^{n} x^{b_i}$$

is a multiple of $(x-1)^k$ if and only if the a_is and b_is form a degree k multigrade. They also had applications to estimating logarithms before modern computers [1], and more recently there have been applications in construcing differences of sums of square roots that are very close to 0, shedding light on the types of root seperation bounds that are possible [2].

Parametric solutions to the multigrade problem have been found for $k = 1, ..., 7$, and an elliptic curve of solutions to the $k = 9$ case was found by Letac in the 1940s. There are also several known solutions to the $k = 8$ and $k = 11$ cases. Chen Shuwen has collected most of the known results online in his webpage [3]. Our goal is to find parametric solutions for larger k.

2 Approach

Let X be the variety of ideal multigrades of degree k (in other words, the multigrades for which n, the number of variables in each set, is $k+1$). We identify a hyperplane of trivial multigrades with $\mathbb{P}^k$. Let C be any curve contained in that hyperplane. Then we have the inclusions

$$C \subset \mathbb{P}^k \subset X \subset \mathbb{P}^{2n-1},$$

and if we let R be the ring $\mathbb{Q}[x_0, ..., x_k, y_0, ..., x_k]$, I be the ideal of C in R, H the ideal of the hyperplane of trivial solutions, and J be the ideal of X, we have the reverse inclusions

$$R \supset I \supset H \supset J.$$

We want to study the deformations of C. According to [4], the space of first order deformations can be identified with the global sections of the normal sheaf to C. For the deformations that remain in $\mathbb{P}^k$, the normal sheaf is $\mathcal{N}_{C/\mathbb{P}^k} = \widetilde{\mathrm{Hom}(I/H, R/I)}$, while for deformations that remain in X the normal sheaf is $\mathcal{N}_{C/X} = \widetilde{\mathrm{Hom}(I/J, R/I)}$. Thus, if $\dim H^0(\mathcal{N}_{C/X}) > \dim H^0(\mathcal{N}_{C/\mathbb{P}^k})$, then the curve C has a nontrivial first order deformation. Hopefully, this first order deformation could then be extended to a full deformation, finally giving us a nontrivial rational curve contained in X.

3 Results

If we restrict to the case where C is a rational curve defined by a homogenous degree d map $(s,t) \to (f_0(s,t), ..., f_k(s,t); f_0(s,t), ..., f_k(s,t))$ (i.e., $x_i = y_i =$

$f_i(s,t))$, then as long as no two f_i are identical, we will show that C has a nontrivial first order deformation within the variety X if and only if

$$\deg \gcd_i \left(\prod_{a<b, a,b\neq i} (f_a(s,t) - f_b(s,t)) \right) \geq d\frac{n(n-3)}{2}.$$

Thus, for $n \leq 3$ all rational curves deform, while for $n > 3$ only special curves can deform. One such curve is the line $(s,t) \to (s+c_0t, s+c_1t, ..., s+c_kt; s+c_0t, ..., s+c_kt)$, where $c_0, ..., c_k$ are arbitrary distinct constants. One family of deformations is given by the family of lines

$$(s,t) \to (s+a_0t, ..., s+a_kt; s+b_0t, ..., s+b_kt),$$

where $(a_0, ..., a_k; b_0, ..., b_k)$ can be any multigrade - in other words, every point of X has a line going through it. This has been known for quite some time, though, and such curves are not seen as providing infinitely many "genuinely different" solutions to the multigrade problem.

A second curve is the "rational normal curve" of degree k defined by clearing denominators in the map $(s,t) \to (\frac{1}{s+c_0t}, ..., \frac{1}{s+c_kt}; \frac{1}{s+c_0t}, ..., \frac{1}{s+c_kt})$ for any distinct $c_0, ..., c_k$. The gcd from the left hand side of our condition is

$$t^{\frac{(n-1)(n-2)}{2}} \prod_i (s+c_it)^{\frac{(n-2)(n-3)}{2}},$$

which has degree

$$\frac{(n-1)(n-2)}{2} + n\frac{(n-2)(n-3)}{2} = 1 + k\frac{n(n-3)}{2},$$

so such a curve always has nontrivial first order deformations! In fact, if we choose $c_i = i$ then we can explicitly calculate a nontrivial second order deformation, given by clearing denominators in the equations:

$$x_i = \frac{1}{s+it} + \epsilon \left(\sum_j (-1)^{i+j} (i-j)^{k-1} \binom{k}{i} \binom{k}{j} \frac{1}{s+jt} \right)$$

$$y_i = \frac{1}{s+it} - \epsilon \left(\sum_j (-1)^{i+j} (i-j)^{k-1} \binom{k}{i} \binom{k}{j} \frac{1}{s+jt} \right)$$

Unfortunately, there is no guarantee that this curve can be deformed "all the way." We would be able to guarantee this if the relative Hilbert scheme of curves contained in X was smooth at C, but unfortunately this is impossible for $n > 3$ - if it was smooth, then the dimension of the tangent space would not decrease as we deformed C into a nonspecial trivial curve, which is a contradiction, since such a curve has no nontrivial deformations.

4 Calculations

We can think of C as a curve in $\mathbb{Q}[x_0, ..., x_k]$ by forgetting about the y coordinates in our trivial hyperplane. Then we get an alternate ideal $I' \approx I/H$ describing C, and the ideal I is the direct sum of H and I'. Thus,

$$\begin{aligned}\mathrm{Hom}(\widetilde{I/J}, R/I) =& \mathrm{Hom}(\widetilde{(I' \oplus H)/J}, R/I) \\ =& \mathrm{Hom}(\widetilde{I/H}, R/I) \oplus \mathrm{Hom}(\widetilde{H/J}, R/I),\end{aligned}$$

so if $\dim H^0(\mathrm{Hom}(\widetilde{H/J}, R/I)) > 0$ then our curve has first order deformations. From now on, we make the assumption that the curve C we are working with is a rational curve abstractly isomorphic to $\mathbb{P}^1$. Then the ring R/I will be a PID, so $\mathrm{Hom}(-, R/I)$ will be finitely generated and torsion free, and thus free. Thus, $\mathrm{Hom}(\widetilde{H/J}, R/I)$ is a locally free sheaf on C. By a theorem of Grothendieck, any locally free sheaf on $\mathbb{P}^1$ can be written as a direct sum of twisted structure sheafs - in this case, $\mathrm{Hom}(\widetilde{H/J}, R/I)$ has rank one, so it must be isomorphic to $\mathcal{O}_C(l)$, where l is the degree of $\mathrm{Hom}(\widetilde{H/J}, R/I) = \mathcal{N}_{C/X}/\mathcal{N}_{C/\mathbb{P}^k}$. Thus, C will have nontrivial first order deformations iff $l \geq 0$.

In order to compute the degree l, we will need to use properties of the cotangent complex. I will follow the notation of Lichtenbaum and Schlessinger [5], in which it is proven that for any map of schemes $Z \to Y \to X$ and any sheaf $\mathcal{F}$ there is a long exact sequence

$$\begin{aligned}0 \to& T^0(Z/Y, \mathcal{F}) \to T^0(Z/X, \mathcal{F}) \to T^0(Y/X, \mathcal{F}) \\ \to& T^1(Z/Y, \mathcal{F}) \to T^1(Z/X, \mathcal{F}) \to T^1(Y/X, \mathcal{F})\end{aligned} \tag{1}$$

If we apply this to the sequence of schemes $C \to X \to \mathrm{Spec}\ \mathbb{Q}$ and the sheaf $\mathcal{O}_C$, we get a short exact sequence of sheaves on C:

$$0 \to 0 \to \mathcal{T}_C \to \mathcal{T}_X \otimes \mathcal{O}_C \to \mathcal{N}_{C/X} \to 0$$

Here, $T^0(C/X, \mathcal{O}_C)$ is 0 since the map $C \to X$ is an embedding, and $T^1(C/\mathrm{Spec}\ \mathbb{Q}, \mathcal{O}_C)$ is 0 since C is a nonsingular curve. If we take the similar short exact sequence corresponding to the sequence of schemes $C \to \mathbb{P}^k \to \text{Spec } \mathbb{Q}$, then we get an exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_C & \longrightarrow & \mathcal{T}_{\mathbb{P}^k} \otimes \mathcal{O}_C & \longrightarrow & \mathcal{N}_{C/\mathbb{P}^k} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_C & \longrightarrow & \mathcal{T}_X \otimes \mathcal{O}_C & \longrightarrow & \mathcal{N}_{C/X} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{T}_X/\mathcal{T}_{\mathbb{P}^k} & \longrightarrow & \mathcal{N}_{C/X}/\mathcal{N}_{C/\mathbb{P}^k} & \longrightarrow & 0
\end{array}$$

The last row is exact by the Snake Lemma. We want to compute the degree of the sheaf on the bottom right. Since degrees are additive along exact sequences, this means that we only have to compute the degrees of $\mathcal{T}_X \otimes \mathcal{O}_C$ and $\mathcal{T}_{\mathbb{P}^k} \otimes \mathcal{O}_C$ to compute l. To compute the latter, we use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^k} \to \mathcal{O}_{\mathbb{P}^k}(1)^n \to \mathcal{T}_{\mathbb{P}^k} \to 0$$

to get $\deg \mathcal{T}_{\mathbb{P}^k} \otimes \mathcal{O}_C = nd$, where d is the degree of the curve C. To compute the degree of $\mathcal{T}_X \otimes \mathcal{O}_C$, we have to use the sequence (1) applied to the sequence of schemes $X \to \mathbb{P}^{2n-1} \to \text{Spec } \mathbb{Q}$:

$$0 \to \mathcal{T}_X \otimes \mathcal{O}_C \to \mathcal{T}_{\mathbb{P}^{2n-1}} \otimes \mathcal{O}_C \to \mathcal{N}_{X/\mathbb{P}^{2n-1}} \otimes \mathcal{O}_C \to T^1(X/\text{Spec } \mathbb{Q}, \mathcal{O}_C) \to 0.$$

Here $T^1(X/\text{Spec } \mathbb{Q}, \mathcal{O}_C)$ may be nonzero since C might pass through the singular locus of X.

The ideal J defining X is generated by the differences of the first k elementary symmetric functions of the x_is and y_is, which are algebraicly independent (so X is a complete intersection). Thus, the graded module J/J^2 is free, with generators of degrees $1, ..., k$, so $\widetilde{J/J^2} = \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-k)$. Taking the dual, we get $\mathcal{N}_{X/\mathbb{P}^{2n-1}} = \text{Hom}(\widetilde{J/J^2}, \mathcal{O}_X) = \mathcal{O}_X(1) \oplus \cdots \oplus \mathcal{O}_X(k)$, so $\deg \mathcal{N}_{X/\mathbb{P}^{2n-1}} \otimes \mathcal{O}_C = d\frac{nk}{2}$.

We can also calculate that $\deg \mathcal{T}_{\mathbb{P}^{2n-1}} \otimes \mathcal{O}_C = 2nd$. Putting it all together, we get the formula

$$\begin{aligned}
l &= -nd + 2nd - d\frac{nk}{2} + \deg T^1(X/\text{Spec } \mathbb{Q}, \mathcal{O}_C) \\
&= \deg T^1(X/\text{Spec } \mathbb{Q}, \mathcal{O}_C) - d\frac{n(n-3)}{2}.
\end{aligned}$$

According to Lemma (3.1.2) of [5], we can explicitly describe $T^1(X/\text{Spec } \mathbb{Q}, \mathcal{O}_C)$. To be as explicit as possible, we will assume the C is given by a degree d map $(s, t) \to (f_0(s, t), f_1(s, t), ..., f_k(s, t); f_0(s, t), ..., f_k(s, t))$ to $\mathbb{P}^{2n-1}$. Then

$T^1(X/\text{Spec } \mathbb{Q}, \mathcal{O}_C)$ is the cokernel of the map from $\text{Hom}(\Omega_{\mathbb{P}^{2n-1}}, \mathcal{O}_C)$ to $\text{Hom}(\widetilde{J/J^2}, \mathcal{O}_C) = \mathcal{N}_{X/\mathbb{P}^{2n-1}} \otimes \mathcal{O}_C$, where the map is left multiplication by the Jacobian matrix of the generators of J, with all the entries substituted for the functions $f_i(s,t)$. Explicitly, this matrix is given by the Vandermonde matrix

$$\begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ 2f_0(s,t) & \dots & 2f_k(s,t) & 2f_0(s,t) & \dots & 2f_k(s,t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ kf_0^{k-1}(s,t) & \dots & kf_k^{k-1}(s,t) & kf_0^{k-1}(s,t) & \dots & kf_k^{k-1}(s,t) \end{pmatrix}$$

At this point we have to make the assumption that no two f_is are identical, so the image of this map has full rank within $\mathcal{N}_{X/\mathbb{P}^{2n-1}} \otimes \mathcal{O}_C$, making the cokernel of this map a torsion sheaf. The degree of a torsion sheaf $\mathcal{F}$ is given in page 149 of Hartshorne [6] as

$$\sum_{P \in C} \text{length}(\mathcal{F}_P),$$

so we just have to compute this length at the various points P of C. In order to accomplish this, it is useful to put our matrix in Smith Normal Form, giving us a matrix

$$\begin{pmatrix} a_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_k & \dots & 0 \end{pmatrix}$$

To compute the length of $\mathcal{F}_P$ at a point P with coordinates (a,b), we can localize the elements of this matrix by the ideal $(bs - at)$ so that the matrix simply multiplies each coordinate by a unit times a power of $(bs - at)$. Then the length of the cokernel of the localized matrix is obviously just the total power of $(bs - at)$ occuring among the a_is. Summing over all points P, we see that the degree of $\mathcal{F}$ is equal to the degree of the polynomial $a_1 a_2 \cdots a_k$.

There is a way to calculate $a_1 a_2 \cdots a_k$ without calculating the Smith Normal Form of our matrix - note that $a_1 a_2 \cdots a_k$ is the gcd of the determinants of all k by k submatrices of our matrix formed by deleting $n+1$ columns. Since this gcd remains unchanged by elementary row and column operations on our matrix, this means we can calculate this gcd on our original matrix to get the same result. Since our original matrix was a Vandermonde matrix, the determinant of the submatrix formed by selecting columns $i_1, ..., i_k$ is just

$$\prod_{a<b}(f_{i_a} - f_{i_b}).$$

Thus, we have proved the following theorem:

Theorem 1. *For any rational curve C defined as the image of a degree d map $(s,t) \to (f_0(s,t), ..., f_k(s,t); f_0(s,t), ..., f_k(s,t))$, $f_i \neq f_j$, C has a nontrivial first order deformation within the variety X if and only if*

$$\deg \gcd_i \left(\prod_{a<b, a,b \neq i} (f_a(s,t) - f_b(s,t)) \right) \geq d \frac{n(n-3)}{2}.$$

Acknowldegements. I would like to thank my mentor, Matthias Flach, for all of his ideas, advice, and support throughout the project. I also would like to acknowledge the creators of *Macaulay 2* for allowing me to test various conjectures involving quite abstract mathematics. Finally, I want to thank the SURF program as well as the Paul K. and Evalyn Cook Richter Memorial Funds for giving me the opportunity to work on this problem.

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