Minimal Taylor Algebras

Zarathustra Brady

Taylor algebras

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  An idempotent algebra is *Taylor* if the variety it generates does not contain a two element set.

  - All algebras in this talk will be idempotent, so I won't mention idempotence further.

Useful facts about Taylor algebras

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  A finite algebra $\mathbb{A}$ is Taylor iff for every number n such that every prime factor of n is greater than $|\mathbb{A}|$, there is an n-ary cyclic term c, i.e.

$$c(x_1, x_2, ..., x_n) \approx c(x_2, ..., x_n, x_1).$$

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  $$c(x_1, x_2, ..., x_n) \approx c(x_2, ..., x_n, x_1).$$

- Corollary

  A finite algebra is Taylor iff it has a 4-ary term t satisfying the identity

  $$t(x, x, y, z) \approx t(y, z, z, x).$$

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- Proposition
  Every finite Taylor algebra has a reduct which is a minimal Taylor algebra.

- Proof.
  There are only finitely many 4-ary terms t which satisfy $t(x, x, y, z) \approx t(y, z, z, x)$. $\square$

First hints of a nice theory

- Theorem
  If $\mathbb{A}$ *is a minimal Taylor algebra,* $\mathbb{B} \in HSP(\mathbb{A})$, $S \subseteq \mathbb{B}$, *and* t *a term of* $\mathbb{A}$ *satisfy*
    - S *is closed under* t,
    - (S, t) *is a Taylor algebra,*

  then S *is a subalgebra of* $\mathbb{B}$, *and is also a minimal Taylor algebra.*

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    - Then

$$f = c(u(x_1, x_2, ..., x_p), u(x_2, x_3, ..., x_1), ..., u(x_p, x_1, ..., x_{p-1}))$$

is a cyclic term of $\mathbb{A}$.

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    - Choose p a prime bigger than $|\mathbb{A}|$ and $|S|$.
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      is a cyclic term of $\mathbb{A}$.
    - Have $f|_S = u|_S$ by idempotence.

A few consequences

- Proposition

  For $\mathbb{A}$ *minimal Taylor,* $a, b \in \mathbb{A}$*, then* $\{a, b\}$ *is a semilattice subalgebra of* $\mathbb{A}$ *with absorbing element* b *iff*

$$\begin{bmatrix} b \\ b \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^2} \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \right\}.$$

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- Proposition

  For $\mathbb{A}$ minimal Taylor, $a, b \in \mathbb{A}$, then $\{a, b\}$ is a majority subalgebra of $\mathbb{A}$ iff

$$\begin{bmatrix} a & b \\ a & b \\ a & b \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^{3\times 2}} \left\{ \begin{bmatrix} a & b \\ a & b \\ b & a \end{bmatrix}, \begin{bmatrix} a & b \\ b & a \\ a & b \end{bmatrix}, \begin{bmatrix} b & a \\ a & b \\ a & b \end{bmatrix} \right\}.$$

A few consequences, ctd.

- Proposition

  For $\mathbb{A}$ minimal Taylor, $a, b \in \mathbb{A}$, then $\{a, b\}$ is a $\mathbb{Z}/2^{\mathrm{aff}}$ subalgebra of $\mathbb{A}$ iff

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  - If there is an automorphism of $\mathbb{A}$ which interchanges a, b, then we only have to consider

$$\mathrm{Sg}_{\mathbb{A}^3} \left\{ \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\}.$$

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- For any $a < \frac{p}{2}$, can make a ternary term $w(x, y, z)$ via:

$$w(x, y, z) = c(\underbrace{x, ..., x}_{a}, \underbrace{y, ..., y}_{p-2a}, \underbrace{z, ..., z}_{a}).$$

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- This satisfies

$$w(x, x, y) \approx w(y, x, x).$$

- Also have

$$w(x, y, x) = c(\underbrace{x, \dots, x}_{a}, \underbrace{y, \dots, y}_{p-2a}, \underbrace{x, \dots, x}_{a}).$$

Daisy Chain Terms, ctd.

- From a sequence

$$a, p-2a, p-2(p-2a), \ldots$$

we get a sequence of ternary terms:

$$\begin{aligned} w_0(x,x,y) \approx w_0(y,x,x) \approx w_1(x,y,x), \\ w_1(x,x,y) \approx w_1(y,x,x) \approx w_2(x,y,x), \\ \vdots \end{aligned}$$

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- If p is large enough and a is close enough to $\frac{p}{3}$, then the sequence can become arbitrarily long.
- Since there are only finitely many ternary functions in $\mathrm{Clo}(\mathbb{A})$, we eventually get a cycle.

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$$\mathbb{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^3} \left\{ \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\} \right\}.$$

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- Proposition

  If $\mathbb{A}$ has daisy chain terms and $a, b \in \mathbb{A}$, then if we consider $\mathbb{D}_{ab}$ as a digraph, it must contain a directed cycle.

Describing a minimal Taylor algebra

- If $p = w_i, q = w_{i+1}$ are any pair of adjacent daisy chain terms, then they satisfy the system

$$\begin{aligned} p(x,x,y) &\approx p(y,x,x) \approx q(x,y,x), \\ q(x,x,y) &\approx q(y,x,x). \end{aligned}$$

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- In particular, the number of minimal Taylor clones on a set of n elements is at most n^{2n^3}.

- Conjecture

  Every minimal Taylor clone can be generated by a *single* ternary function.

Daisy chain terms in the basic algebras

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  If w_i are daisy chain terms and $\mathbb{A}$ is a semilattice, then each w_i is the symmetric ternary semilattice operation on $\mathbb{A}$.

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- Proposition

  If w_i are daisy chain terms and $\mathbb{A}$ is affine, then there is a sequence a_i such that w_i is given by

  $$w_i(x, y, z) = a_i x + (1 - 2a_i)y + a_i z,$$

  with $a_{i+1} = 1 - 2a_i$.
  If $a_0 = 0$, then w_1 is the Mal'cev operation $x - y + z$ and w_{-1} is the operation $\frac{x+z}{2}$.

Bulatov's graph

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- Definition

  If $\mathbb{A}$ is minimal Taylor and $a, b \in \mathbb{A}$, then (a, b) is an *edge* if there is a congruence θ on $\mathrm{Sg}\{a, b\}$ s.t.

  $$\mathrm{Sg}\{a, b\}/\theta$$

  is isomorphic to either a two-element semilattice, a two element majority algebra, or an affine algebra.

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    - $\mathbb{A}$ has no proper congruences.
  - It's not hard to show there must be an automorphism interchanging a, b.
  - Consider the binary relation $\mathbb{D}_{ab}$!

Connectivity, ctd.

- Recall the definition of $\mathbb{D}_{ab}$:

$$\mathbb{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^3} \left\{ \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\} \right\}.$$

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- The daisy chain terms give us $c, d, e \in \mathbb{A}$ such that

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- If both $\mathrm{Sg}\{a, d\}$ and $\mathrm{Sg}\{d, b\}$ are proper subalgebras, then the hypergraph of proper subalgebras is connected.
- Then we can show $\mathbb{D}_{ab}$ is subdirect, and the proof flows naturally from here.

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  For every semilattice edge from a to b, there is a b' in the congruence class of b such that $\{a, b'\}$ is a two element semilattice algebra.

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  - Similar statements fail for majority edges and affine edges.
  - There are minimal Taylor algebras $\mathbb{A}, \mathbb{B}$ of size 4 which have congruences θ such that:
    - $\mathbb{A}/\theta$ is a two element majority algebra and $\mathbb{B}/\theta$ is $\mathbb{Z}/2^{aff}$,
    - each congruence class of θ is a copy of $\mathbb{Z}/2^{aff}$,
    - every proper subalgebra of $\mathbb{A}$ or $\mathbb{B}$ is contained in a congruence class of θ,
    - $\mathbb{A}$ has a 3-edge term and $\mathbb{B}$ is Mal'cev,
    - θ is the center of $\mathbb{A}$ or $\mathbb{B}$ in the sense of commutator theory.

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- g commutes with the cyclic permutation $\sigma = (a\ b\ c\ d)$ and satisfies

$$\begin{aligned} g(a, a, b) &= a, \\ g(a, a, c) &= c, \\ g(a, a, d) &= c, \\ g(a, b, c) &= c. \end{aligned}$$

Evil algebra #1

- $\mathbb{A} = (\{a, b, c, d\}, g)$, where g is an idempotent ternary symmetric operation.
- g commutes with the cyclic permutation $\sigma = (a\ b\ c\ d)$ and satisfies

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- The algebra $\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$ has a congruence ψ corresponding to the partition

$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ a \end{bmatrix} \right\}, \left\{ \begin{bmatrix} a \\ d \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}, \begin{bmatrix} c \\ b \end{bmatrix}, \begin{bmatrix} d \\ c \end{bmatrix} \right\},$$

such that $\mathbb{S}/\psi$ is isomorphic to $\mathbb{Z}/2^{aff}$.

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- The polynomials $+_a = p(\cdot, a, \cdot), +_b = p(\cdot, b, \cdot)$ define abelian groups:

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b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

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Zhuk's four cases

- Theorem (Zhuk)

  If $\mathbb{A}$ is minimal Taylor, then at least one of the following holds:
    - *$\mathbb{A}$ has a proper binary absorbing subalgebra,*
    - *$\mathbb{A}$ has a proper "center",*
    - *$\mathbb{A}$ has a nontrivial affine quotient, or*
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- Definition

  $\mathbb{C} \leq \mathbb{A}$ is a *center* of $\mathbb{A}$ if there exist
  - a binary-absorption-free Taylor algebra $\mathbb{B}$ and
  - a subdirect relation $\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B}$, such that
  - $\mathbb{C} = \left\{ c \in \mathbb{A} \text{ s.t. } \forall b \in \mathbb{B}, \begin{bmatrix} c \\ b \end{bmatrix} \in \mathbb{R} \right\}$.

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- Theorem (Zhuk)

  If $\mathbb{C}$ is a center of $\mathbb{A}$, then $\mathbb{C}$ is a ternary absorbing subalgebra of $\mathbb{A}$.

Centers and Daisy Chain terms

Theorem

If $\mathbb{A}$ is minimal Taylor and $\mathbb{M} \in HSP(\mathbb{A})$ is the two element majority algebra on the domain $\{0,1\}$, then the following are equivalent:

- *$\mathbb{C}$ is a ternary absorbing subalgebra of $\mathbb{A}$,*
- *there is a p-ary cyclic term c of $\mathbb{A}$ such that whenever $\#\{x_i \in \mathbb{C}\} > \frac{p}{2}$, we have*

$$c(x_1, ..., x_p) \in \mathbb{C},$$

- *the binary relation $\mathbb{R} \subseteq \mathbb{A} \times \mathbb{M}$ given by*

$$\mathbb{R} = (\mathbb{A} \times \{0\}) \cup (\mathbb{C} \times \{0,1\})$$

  is a subalgebra of $\mathbb{A} \times \mathbb{M}$,
- *every daisy chain term $w_i(x,y,z)$ witnesses the fact that $\mathbb{C}$ ternary absorbs $\mathbb{A}$.*

Centers produce majority quotients

- If $\mathbb{C}, \mathbb{D}$ are centers, then for any daisy chain terms w_i, we must have

$$w_i(\mathbb{C}, \mathbb{C}, \mathbb{D}), w_i(\mathbb{C}, \mathbb{D}, \mathbb{C}), w_i(\mathbb{D}, \mathbb{C}, \mathbb{C}) \subseteq \mathbb{C}$$

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  so $\mathbb{C} \cup \mathbb{D}$ is a subalgebra of $\mathbb{A}$.

- If $\mathbb{C} \cap \mathbb{D} = \emptyset$, then the equivalence relation θ on $\mathbb{C} \cup \mathbb{D}$ with parts $\mathbb{C}, \mathbb{D}$ is preserved by each daisy chain term w_i, and $(\mathbb{C} \cup \mathbb{D})/\theta$ is a two element majority algebra.

Binary absorption is strong absorption

Theorem

If $\mathbb{A}$ is minimal Taylor, then the following are equivalent:

- $\mathbb{B}$ *binary absorbs* $\mathbb{A}$,
- *there exists a cyclic term* c *such that if any* $x_i \in \mathbb{B}$, *then* $c(x_1, \ldots, x_p) \in \mathbb{B}$,
- *the ternary relation*

$$\mathbb{R} = \{(x, y, z) \text{ s.t. } (x \notin \mathbb{B}) \implies (y = z)\}$$

  is a subalgebra of $\mathbb{A}^3$,
- *every term* f *of* $\mathbb{A}$ *which depends on all its inputs is such that if any* $x_i \in \mathbb{B}$, *then* $f(x_1, \ldots, x_n) \in \mathbb{B}$.

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  If $\mathbb{A}$ is minimal Taylor and $\mathbb{A} = \mathrm{Sg}\{a, b\}$, then the following are equivalent:
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  - Minimal Taylor algebras generated by two elements are nicer than general minimal Taylor algebras.
  - It's good enough to understand such algebras.

Big conjecture

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  Suppose $\mathbb{A}$ is minimal Taylor, generated by two elements a, b, and has no affine or semilattice quotient. Then each of a, b is contained in a proper ternary absorbing subalgebra of $\mathbb{A}$.

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- Theorem (Kearnes, Szendrei)

  Suppose a minimal Taylor algebra has no semilattice edges and has its clone generated by a single ternary term. Then it has a 3-edge term.

Thank you for your attention.