COARSE CLASSIFICATION OF BINARY MINIMAL CLONES

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1. Introduction

The classification of minimal clones on a finite set is a very old problem, studied by many authors. For a survey of previous results, the reader is directed to the papers of Quackenbush [10] and Csákány [4]. First we review a few of the basic definitions and results.

Definition 1. A clone $\mathcal{C}$ is *minimal* if $f \in \mathcal{C}$ nontrivial implies $\mathcal{C} = \mathrm{Clo}(f)$. An algebra $\mathbb{A}$ will be called *clone-minimal* (equivalently: $\mathrm{Clo}(\mathbb{A})$ is a *minimal clone*) if $\mathbb{A}$ has no nontrivial proper reduct.

Proposition 1. *If* $\mathrm{Clo}(f)$ *is minimal and* $g \in \mathrm{Clo}(f)$ *nontrivial, then* $f \in \mathrm{Clo}(g)$.

Definition 2. $\mathbb{A}$ is called a *set* if all of its operations are projections. Otherwise, we say $\mathbb{A}$ is *nontrivial.*

Proposition 2. *If* $\mathrm{Clo}(\mathbb{A})$ *is minimal and* $\mathbb{B} \in \mathrm{Var}(\mathbb{A})$ *nontrivial, then* $\mathrm{Clo}(\mathbb{B})$ *is minimal.*

Theorem 1 (Rosenberg [11])**.** *Suppose that* $\mathbb{A} = (A, f)$ *is a finite clone-minimal algebra, and* f *has minimal arity among nontrivial elements of* $\mathrm{Clo}(\mathbb{A})$*. Then one of the following is true:*

(1) f *is a unary operation which is either a permutation of prime order or satisfies* $f(f(x)) \approx f(x)$,
(2) f *is ternary, and* $\mathbb{A}$ *is the idempotent reduct of a vector space over* $\mathbb{F}_2$,
(3) f *is a ternary majority operation,*
(4) f *is a semiprojection of arity at least* 3,
(5) f *is an idempotent binary operation.*

Rosenberg's classification is not fully satisfactory: it can be very difficult to check whether a given majority operation, semiprojection, or idempotent binary operation generates a minimal clone. This paper is mainly concerned with the binary case. Previous authors have proven classifications of binary minimal clones under additional assumptions, such as the assumption that the binary operation is *entropic* [7], or the assumption that the associated algebra is *weakly abelian* [12], or the assumption that the number of binary operations in the clone is small [8]. In this paper, we consider the general case of binary minimal clones.

Definition 3. We say a property $\mathcal{P}$ of functions f is *nice* if it satisfies the following conditions:

- Given f as input, we can verify in polynomial time whether f has property $\mathcal{P}$,
- If f has property $\mathcal{P}$ and $g \in \mathrm{Clo}(f)$ is nontrivial, then there is a nontrivial $f' \in \mathrm{Clo}(g)$ such that f' has property $\mathcal{P}$.
- There exists a fixed nontrivial algebra $\mathbb{A}_{\mathcal{P}}$ such that for any nontrivial algebra $\mathbb{A} = (A, f)$ where f has the property $\mathcal{P}$, the algebra $\mathbb{A}_{\mathcal{P}}$ is in the variety generated by $\mathbb{A}$.
- The set of finite algebras (A, f) such that f has the property $\mathcal{P}$ is closed under finite products, homomorphic images, and subalgebras, that is, it forms a pseudovariety.

The first three cases in Rosenberg's classification are given by nice properties (with the caveat that in the case of unary permutations of prime order, we actually get an infinite family of nice properties indexed by the primes). As an example, we'll check that being a ternary majority operation is a nice property. The first, third, and fourth conditions are straightforward to verify. For the second condition, we use the following elementary result, which appears in [14] and is used there to simplify the study of minimal majority clones.

Proposition 3. *If f is a majority operation and $g \in \mathrm{Clo}(f)$ is nontrivial, then g is a near-unanimity operation. In this case, g has a majority term as an identification minor.*

Proof. The proof is by induction on the construction of g in terms of f. If $g = f(g_1, g_2, g_3)$, then by induction each g_i is either a near-unanimity term or a projection. If two of the g_is are equal to the same projection, then so is g. Otherwise, it is easy to check that g is a near-unanimity operation.

Since any near-unanimity operation is not a semiprojection, if g is not already majority we can identify two variables to get a nontrivial $g' \in \mathrm{Clo}(g)$ of smaller arity. Thus g has a majority term as an identification minor. □

The fact that being a majority operation is nice implies that in order to check whether a majority clone $\mathrm{Clo}(f)$ is minimal, one only needs to enumerate the ternary majority operations $g \in \mathrm{Clo}(f)$ and check that $f \in \mathrm{Clo}(g)$ for each one. While this may be difficult, we are at the very least assured that if f is a majority operation, then $\mathrm{Clo}(f)$ contains *some* minimal majority clone (in fact every minimal clone contained in $\mathrm{Clo}(f)$ will be a majority clone).

On the other hand, if one is given a binary idempotent operation f, then it can difficult to rule out the possibility that $\mathrm{Clo}(f)$ might contain a semiprojection of very large arity. As a result, checking whether a binary operation generates a minimal clone could in principle be an enormous undertaking. The goal of the present paper is to provide a *coarse classification* of binary minimal clones, that is, a list $\mathcal{P}_1, \mathcal{P}_2, ...$ of nice properties such that every binary minimal clone contains an operation satisfying exactly one of the nice properties $\mathcal{P}_i$.

As a starting point, a previous paper by the present author has given a coarse classification of minimal clones which have a Taylor operation, into just three cases. Two of these cases are defined by nice properties, while the third (vector spaces over a prime field) is given in terms of an infinite family of nice properties, one for each prime p.

Theorem 2 (Clone-minimal Taylor algebras [2])**.** *Suppose $\mathbb{A}$ is a finite algebra which is both clone-minimal and Taylor. Then one of the following is true:*

(1) *$\mathbb{A}$ is the idempotent reduct of a vector space over $\mathbb{F}_p$ for some prime p,*
(2) *$\mathbb{A}$ is a majority algebra,*
(3) *$\mathbb{A}$ is a spiral.*

The last case above was one of the author's main motivations for introducing the concept of nice properties. Spirals are defined as follows.

Definition 4. An algebra $\mathbb{A} = (A, f)$ is a spiral if f is binary, idempotent, commutative, and for any $a, b \in \mathbb{A}$ either $\{a, b\}$ is a subalgebra of $\mathbb{A}$, or $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$ has a surjective map to the free semilattice on two generators.

The reader may find it instructive to check that being a spiral is a nice property. While this follows from the results of [2], it is simple enough to give a direct argument. For now, we just note that it is possible to test whether a given algebra $\mathbb{A} = (A, f)$ is a spiral in time polynomial in $|A|$: for any a, b with $\{a, b\}$ not a subalgebra of $\mathbb{A}$, if $\mathrm{Sg}_{\mathbb{A}}\{a, b\} = \{a, b\} \cup S$ with $a, b \notin S$, then any surjective homomorphism from $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$ to the free semilattice on two generators x, y must map a to one generator, say x, map b to the other generator y, and map every element of S to xy. Thus, the existence of such a surjective homomorphism is equivalent to S being a subalgebra of $\mathbb{A}$ such that $aS \subseteq S$ and $bS \subseteq S$.

The main classification result of this paper is the following coarse classification of the non-Taylor binary minimal algebras.

Theorem 3. *Suppose that $\mathbb{A} = (A, f)$ and $\mathrm{Clo}(f)$ is a binary minimal clone which is not Taylor. Then, after possibly replacing $f(x, y)$ by $f(y, x)$, one of the following is true:*

(1) *$\mathbb{A}$ is a rectangular band, i.e. an idempotent groupoid satisfying $(xy)(zw) \approx xw$,*
(2) *$\mathbb{A}$ is a p-cyclic groupoid for some prime p,*
(3) *there is a nontrivial $s \in \mathrm{Clo}(f)$ which is a "partial semilattice operation": $s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y)$,*
(4) *$\mathbb{A}$ is an idempotent groupoid satisfying $(xy)(zx) \approx xy$ (a "meld"),*
(5) *$\mathbb{A}$ is a "dispersive algebra" (defined below).*

Furthermore, each of the above cases is defined by a nice property, other than the case of p-cyclic groupoids, which consists of an infinite family of nice properties indexed by the primes.

Most of the cases appearing in the classification have been described by previous authors. Rectangular bands are well-known in the theory of bands. The structure of p-cyclic groupoids was described by Płonka [9]. Partial semilattice operations were isolated by Bulatov [3] in his study of colored graphs attached to finite Taylor algebras, but they were not named there or studied for their own sake. Melds were described by Lévai and Pálfy in Theorem 5.2(e) of [8], which classifies binary minimal clones having exactly four binary operations, but melds were not given a name there, and no structure theory for them was given.

The last case of the above theorem - the case of dispersive algebras - has a definition which has a similar flavour to the definition of spirals, but is somewhat harder to work with in practice. The author has struggled for some time to give a structure theory for this case, but was ultimately unsuccessful. Several difficult conjectures about the structure of dispersive algebras are given at the end of the paper.

To define the dispersive algebras, we first define the variety $\mathcal{D}$ of idempotent groupoids satisfying

$$x(yx) \approx (xy)x \approx (xy)y \approx (xy)(yx) \approx xy, \tag{$\mathcal{D}1$}$$

$$\forall n \geq 0 \quad x(...((xy_1)y_2) \cdots y_n) \approx x. \tag{$\mathcal{D}2$}$$

This variety of minimal clones appears in Lévai and Pálfy [8], and the notation $\mathcal{D}$ for this variety is from Waldhauser's thesis [13].

Proposition 4 (Lévai, Pálfy [8])**.** *If $\mathbb{A} \in \mathcal{D}$, then $\mathrm{Clo}(\mathbb{A})$ is a minimal clone. Also, $\mathcal{F}_{\mathcal{D}}(x, y)$ has exactly four elements: x, y, xy, yx.*

Definition 5. An idempotent groupoid $\mathbb{A}$ is *dispersive* if it satisfies ($\mathcal{D}2$) and if for all $a, b \in \mathbb{A}$, either $\{a, b\}$ is a two element subalgebra of $\mathbb{A}$ or there is a surjective map

$$\mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x, y).$$

The name "dispersive" for such algebras was chosen in order to reflect the fact that they satisfy very few *absorption identities*. Absorption identities are a crucial tool in the study of minimal clones. (There is an unfortunate naming collision here: absorption identities have nothing to do with the theory of absorbing subalgebras from [1] which recently found many applications in the study of Taylor algebras.)

Definition 6. An *absorption identity* is an identity of the form

$$t(x_1, ..., x_n) \approx x_i.$$

Proposition 5 (Kearnes [6], Lévai, Pálfy [8])**.** *If $\mathbb{A}$ is clone-minimal and $\mathbb{B} \in \mathrm{Var}(\mathbb{A})$ is nontrivial, then any absorption identity which holds in $\mathbb{B}$ must also hold in $\mathbb{A}$.*

Proof. Suppose that the absorption identity $t(x_1, ..., x_n) \approx x_i$ holds in $\mathbb{B}$ but not in $\mathbb{A}$. Then t generates a nontrivial proper subclone of $\mathrm{Clo}(\mathbb{A})$: the reduct of $\mathbb{B}$ with basic operation t is trivial, so it must be proper, and it is nontrivial on $\mathbb{A}$ since t can't act as any projection on $\mathbb{A}$. □

As a consequence of the third condition for being a nice property, if $\mathcal{P}$ is a nice property and f has property $\mathcal{P}$, then the set of absorption identities that hold in $\mathbb{A} = (A, f)$ depends only on $\mathcal{P}$. Thus the coarse classification of binary minimal clones given in this paper is also a classification of the possible collections of absorption identities which can be satisfied in a binary minimal clone.

In the case of partial semilattice operations, no absorption identities hold at all (other than those following from idempotence), for the simple reason that every nontrivial partial semilattice contains a two-element semilattice subalgebra. In the case of melds, every absorption identity follows from idempotence and the identity

$$x((yx)z) \approx x,$$

and in fact this identity is equivalent to the defining identity $(xy)(zx) \approx xy$ of melds (modulo idempotence). In the case of dispersive algebras, all absorption identities follow from idempotence and the absorption identity ($\mathcal{D}2$).

In light of the coarse classification of binary minimal clones provided in this paper, it is natural to ask whether a coarse classification of semiprojections can be found. Such a classification would, in a certain sense, complete the classification of minimal clones which was started by Rosenberg [11]. The author has not made any attempt at classifying semiprojections, but the main result of [5] which classifies *conservative* semiprojections indicates that a classification may be within reach.

2. Proof of the classification

Definition 7. If $t : \mathbb{A}^2 \to \mathbb{A}$ is a binary function and $\mathbb{A}$ is finite (or even profinite), then we define t^∞ to be the pointwise limit

$$t^\infty(x, y) = \lim_{n\to\infty} t^{n!}(x, y),$$

where $t^1 = t$ and $t^{n+1}(x, y) = t(x, t^n(x, y))$.

Proposition 6. *For any binary term t, we have*

$$t^\infty(x, t^\infty(x, y)) \approx t^\infty(x, y).$$

If t is idempotent, then so is t^∞.

Proposition 7. *If f is an idempotent binary term which satisfies the identity*

$$f(x, f(x, y)) \approx f(x, y),$$

and if we define a term u by

$$u(x, y) = f(x, f(y, x)),$$

then u satisfies the identity

$$u(u(x, y), x) \approx u(x, y).$$

Similarly, if an idempotent term g satisfies the identity $g(g(x, y), y) \approx g(x, y)$, then $g(g(x, y), x)$ is a binary term satisfying the above identity.

Proof. We have

$$\begin{aligned} u(u(x,y),x) &\approx f(u(x,y), f(x, u(x,y))) \\ &\approx f(u(x,y), f(x, f(x, f(y,x)))) \\ &\approx f(u(x,y), f(x, f(y,x))) \\ &\approx f(u(x,y), u(x,y)) \\ &\approx u(x,y). \end{aligned}$$

For the last claim, take $f(x,y) = g(y,x)$. □

Proposition 8. *If u is a binary term which satisfies the identity*

$$u(u(x,y),x) \approx u(x,y),$$

then $s = u^\infty$ satisfies the identity

$$s(x, s(x,y)) \approx s(s(x,y),x) \approx s(x,y).$$

Proof. First, note that we have

$$\begin{aligned} u(u(x,y), u(x,y)) &\approx u(u(u(x,y),x), u(x,y)) \\ &\approx u(u(x,y),x) \\ &\approx u(x,y). \end{aligned}$$

Define u^n as in the definition of u^∞. Then for any m we have

$$u^m(u(x,y),x) \approx u(x,y),$$

and on replacing y by $u^{n-1}(x,y)$, we get

$$u^m(u^n(x,y),x) \approx u^n(x,y)$$

for any m, n. □

Definition 8. We say that an idempotent binary operation s is a *partial semilattice* if it satisfies the identity

$$s(x, s(x,y)) \approx s(s(x,y),x) \approx s(x,y).$$

Proposition 9. *If an idempotent algebra $\mathbb{A}$ has $a \neq b \in \mathbb{A}$ with*

$$(b,b) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$$

then it has a nontrivial partial semilattice operation. Furthermore, if s is a partial semilattice operation, then for any nontrivial term $t \in \mathrm{Clo}(s)$ there is a nontrivial partial semilattice operation $s' \in \mathrm{Clo}(t)$.

Proof. Choose $f \in \mathrm{Clo}(\mathbb{A})$ such that $f(a,b) = f(b,a) = b$. Let $u(x,y) = f^\infty(x, f^\infty(y,x))$ and let $s = u^\infty$, then s is a partial semilattice, and we have $s(a,b) = s(b,a) = b$ so s is nontrivial.

For the second part, if s is a nontrivial partial semilattice operation, then there are $a, b \in \mathbb{A}$ with $s(a,b) \neq a$, so $\{a, s(a,b)\}$ is a proper subalgebra of $\mathbb{A}$, on which s acts as a semilattice operation. Then any nontrivial $t \in \mathrm{Clo}(s)$ which depends on all its inputs also acts as a semilattice operation on $\{a, s(a,b)\}$, so if we take $g(x,y) = t(x,y,...,y)$ then we have $g(a, s(a,b)) = g(s(a,b),a) = s(a,b)$, so by the first part of the argument there is a nontrivial partial semilattice operation $s' \in \mathrm{Clo}(g) \subseteq \mathrm{Clo}(t)$. □

Definition 9. We say that an algebra $\mathbb{A}$ is a *set* if all of its basic operations are projections. We say that $\mathbb{A}$ is *nontrivial* if it is not a set.

Proposition 10. *If* $\mathbb{A}$ *is a minimal clone and* $\mathbb{B} \in \mathrm{Var}(\mathbb{A})$ *is nontrivial, then for any term* $t \in \mathrm{Clo}(\mathbb{A})$ *such that the identity*

$$t^{\mathbb{B}}(x_1, ..., x_n) \approx x_1$$

holds in $\mathbb{B}$*, we have* $t(x_1, ..., x_n) \approx x_1$ *in* $\mathbb{A}$ *as well.*

Definition 10. We say that an idempotent binary operation f is a *rectangular band* if it satisfies the identity

$$f(f(x, y), f(z, w)) \approx f(x, w).$$

Proposition 11. *If a minimal clone* $\mathbb{A}$ *has a term* f *such that there are algebras* $\mathbb{B}_1, \mathbb{B}_2 \in \mathrm{Var}(\mathbb{A})$ *such that* $f^{\mathbb{B}_1}$ *is first projection and* $f^{\mathbb{B}_2}$ *is second projection, then* $\mathbb{A}$ *is a rectangular band.*

Proof. Assume without loss of generality that $|\mathbb{B}_1| = |\mathbb{B}_2| = 2$, and let $\mathbb{B} = \mathbb{B}_1 \times \mathbb{B}_2$. Also, assume that f is binary (otherwise replace it with $f(x, y, ..., y)$). On $\mathbb{B}$, f satisfies the identity

$$f(f(f(u, x), y), f(z, f(w, u))) \approx u,$$

so this identity must hold on $\mathbb{A}$ as well. Similarly, we have

$$f(f(x, w), x) \approx x$$

and

$$f(w, f(x, w)) \approx w,$$

so

$$f(f(x, y), f(z, w)) \approx f(f(f(f(x, w), x), y), f(z, f(w, f(x, w)))) \approx f(x, w).$$ □

Definition 11. Let $\mathrm{Clo}_2(\mathbb{A})$ be the set of binary terms of $\mathbb{A}$, and let $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ be the set of binary terms of $\mathbb{A}$ which restrict to the first projection on some algebra $\mathbb{B} \in \mathrm{Var}(\mathbb{A})$ of size at least 2.

Theorem 4. *If* $\mathbb{A}$ *is a binary minimal clone which is not a rectangular band and which does not have any nontrivial partial semilattice operations, then for any* $f, g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ *we have*

$$f(x, g(x, y)) \approx x.$$

If f, g *are nontrivial, then we also have* $f(g(x, y), x) \not\approx x$*, and more generally for any* $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ *we have*

$$f(g(x, y), h(x, y)) \not\approx x.$$

Proof. We may assume $\mathbb{A}$ is not Taylor. Choose $\mathbb{B} \in \mathrm{Var}(\mathbb{A})$ such that $\mathbb{B}$ is not a set, but such that every proper subalgebra or quotient of $\mathbb{B}$ is a set. Suppose for contradiction that there are $a, b \in \mathbb{B}$ such that

$$f(a, g(a, b)) \neq a.$$

Then we have $\mathrm{Sg}_{\mathbb{B}}\{a, g(a, b)\} = \mathbb{B}$, so there is some $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ with

$$h(a, b) = b.$$

The other assertions we need to prove are handled similarly - in each case, if one of them is violated, then we can find $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ and $a, b \in \mathbb{B}$ with $h(a, b) = b$; this is what we will use to produce a contradiction.

Define $\mathbb{S} \leq_{sd} \mathbb{B}^2$ by

$$\mathbb{S} = \mathrm{Sg}_{\mathbb{B}^2}\{(a, b), (b, a)\}.$$

If $\mathbb{S}$ is the graph of an automorphism of $\mathbb{B}$ then we have $h(b, a) = a$, so $\{a, b\}$ is a subalgebra of $\mathbb{B}$ on which h acts like second projection, contradicting the assumption that $\mathbb{A}$ is not a rectangular band.

If $\mathbb{S}$ is not linked, then the linking congruence of $\mathbb{S}$ defines a nontrivial congruence $\sim$ on $\mathbb{B}$. Since $\mathbb{B}/{\sim}$ is a set, we have $b = h(a, b) \sim a$, and since a, b generate $\mathbb{B}$ we have $a \not\sim b$, a contradiction.

Thus $\mathbb{S}$ must be linked. In particular, there must be some proper subalgebra $\mathbb{C} < \mathbb{B}$ such that

$$\pi_2(\mathbb{S} \cap (\mathbb{C} \times \mathbb{B})) = \mathbb{B}.$$

Let $c, d \in \mathbb{C}$ with $(a, c), (b, d) \in \mathbb{S}$. Since $\mathbb{C}$ is a set, we have $h(c, d) = c$, so

$$(b, c) = (h(a, b), h(c, d)) = h((a, c), (b, d)) \in \mathbb{S}.$$

If b, c generate $\mathbb{B}$, then from $(a, b), (a, c) \in \mathbb{S}$ we see that $(a, a) \in \mathbb{S}$, contradicting the assumption that $\mathbb{A}$ has no nontrivial partial semilattice terms. Otherwise $\{b, c\}$ is a set, and we must have $h(b, c) = b$, so

$$(b, b) = (h(a, b), h(b, c)) = h((a, b), (b, c)) \in \mathbb{S},$$

again contradicting the assumption that $\mathbb{A}$ has no nontrivial partial semilattice terms. □

Definition 12. We say that an idempotent binary operation f is a *p-cyclic groupoid* if it satisfies the identities

$$f(x, f(y, z)) \approx f(x, y),$$
$$f(f(x, y), z) \approx f(f(x, z), y),$$

and

$$f(\cdots f(f(x, y), y) \cdots y) \approx x,$$

where there are p ys in the last identity.

Theorem 5. *If a binary minimal clone is not a rectangular band and does not have any nontrivial term f satisfying the identity*

$$f(f(x, y), f(y, x)) \approx f(x, y),$$

then it is a p-cyclic groupoid for some prime p.

We will prove this in a series of lemmas, all using the following assumption:

($*$) $\mathbb{A}$ is a finite binary minimal clone which is not a rectangular band and $\mathbb{A}$ has no nontrivial term satisfying the identity $f(f(x, y), f(y, x)) \approx f(x, y)$.

Definition 13. For any $f, g \in \mathrm{Clo}_2(\mathbb{A})$, define their circular composition $f * g \in \mathrm{Clo}_2(\mathbb{A})$ by

$$(f * g)(x, y) = f(g(x, y), g(y, x)).$$

Proposition 12. *If $\mathbb{A}$ satisfies ($*$), then $\mathrm{Clo}_2(\mathbb{A})$ and $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ form groups under $*$. In particular, for any binary term $f \in \mathrm{Clo}(\mathbb{A})$ there exists a binary term $f^- \in \mathrm{Clo}(\mathbb{A})$ such that*

$$f^-(f(x, y), f(y, x)) \approx f(f^-(x, y), f^-(y, x)) \approx x.$$

Proposition 13. *If $\mathbb{A}$ satisfies ($*$), then $\mathbb{A}$ is not a Taylor algebra and $\mathbb{A}$ has no nontrivial partial semilattice operations.*

Proof. That $\mathbb{A}$ is not Taylor follows from the fact that every binary minimal clone which is also Taylor has a term f such that $f(x, y) \approx f(y, x)$.

Suppose for contradiction that $\mathbb{A}$ has a nontrivial partial semilattice operation s, then by the previous proposition there is a term $s^- \in \mathrm{Clo}(\mathbb{A})$ such that

$$s^-(s(x, y), s(y, x)) \approx x.$$

Replacing y with $s(x, y)$ above, we see that

$$s(x, y) \approx s^-(s(x, y), s(x, y)) \approx x,$$

contradicting the assumption that s is nontrivial. □

Proposition 14. *If $\mathbb{A}$ satisfies ($*$), $\mathbb{B} \in \mathrm{Var}(\mathbb{A})$ is not a set, and $f, g \in \mathrm{Clo}_2(\mathbb{A})$ have $f^{\mathbb{B}} = g^{\mathbb{B}}$, then $f(x, y) \approx g(x, y)$.*

Proof. We have

$$f^-(g(x,y),g(y,x)) = x$$

for any $x, y \in \mathbb{B}$, so by clone minimality this identity also holds in $\mathbb{A}$ (otherwise, $f^-(f,g)$ would generate a strictly smaller nontrivial clone). Thus, we have

$$\begin{aligned} f(x,y) &\approx f(f^-(g(x,y),g(y,x)), f^-(g(y,x),g(x,y))) \\ &\approx g(x,y). \end{aligned}$$

□

Proposition 15. *If $\mathbb{A}$ satisfies $(*)$ and $t \in \mathrm{Clo}_3(\mathbb{A})$, $f \in \mathrm{Clo}_2(\mathbb{A})$ satisfy*

$$t(x,y,z) = f(x,y)$$

whenever two of x, y, z are equal, then $t(x,y,z) \approx f(x,y)$.

Proof. Since $\mathbb{A}$ has no ternary semiprojections, we have the identities

$$f^-(t(x,y,z), f(y,x)) \approx x$$

and

$$f^-(f(y,x), t(x,y,z)) \approx y.$$

Thus we have

$$\begin{aligned} t(x,y,z) &\approx f(f^-(t(x,y,z),f(y,x)), f^-(f(y,x),t(x,y,z))) \\ &\approx f(x,y). \end{aligned}$$

□

Lemma 1. *If $\mathbb{A}$ satisfies $(*)$ and there are nontrivial terms $f, g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ satisfying*

$$f(x, g(y,x)) \approx f(x,y),$$

then $\mathbb{A}$ is a p-cyclic groupoid for some prime p.

Proof. By Theorem 4, we have

$$f(x, g(x,y)) \approx x,$$

so

$$f(x, g(y,z)) = f(x,y)$$

whenever two of x, y, z are equal. Thus, we have

$$f(x, g(y,z)) \approx f(x,y).$$

Since g is nontrivial, for any $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ we have $h \in \mathrm{Clo}(g)$ and by iteratively applying the above identity we deduce that

$$f(x, h(y,z)) \approx f(x,y).$$

Thus, for any $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ we have

$$\begin{aligned} f(h(x,y),x) &\approx f(h(x,y), h^-(h(x,y),h(y,x))) \\ &\approx f(h(x,y),h(x,y)) \\ &\approx h(x,y), \end{aligned}$$

and in particular we have

$$f(f(x,y),z) = f(f(x,z),y)$$

whenever two of x, y, z are equal. Since we have

$$f(f^-(x,y),y) \approx f(f^-(x,y), f^-(y,x)) \approx x$$

and

$$f^-(f(x,y),y) \approx f^-(f(x,y),f(y,x)) \approx x$$

(using here that $f^- = f^{*n}$ for some n - alternatively, we could deduce that $f^-(f(x,y),y) \approx x$ from $f(f^-(x,y),y) \approx x$ from the finiteness of $\mathcal{F}_{\mathbb{A}}(x,y)$), we see that

$$f^-(f^-(f(f(x,y),z),y),z) = x$$

whenever two of x, y, z are equal. Since $\mathbb{A}$ has no nontrivial semiprojections, the above holds identically, and so we get

$$f(f(x,y),z) \approx f(f(x,z),y).$$

Now define a sequence of functions f_n by $f_0 = \pi_1$, $f_1 = f$, and

$$f_{n+1}(x,y) = f(f_n(x,y),y).$$

We see that every element of $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ can be written as f_n for some n. Letting $p = |\mathrm{Clo}_2^{\pi_1}(\mathbb{A})|$, we see that $f^- \approx f_{p-1}$, $f_p(x,y) \approx x$, and

$$f_i(f_j(x,y),y) \approx f_{i+j}(x,y).$$

If d is a nontrivial divisor of p then f_d is nontrivial and $f \notin \mathrm{Clo}(f_d)$, contradicting the assumption that $\mathbb{A}$ is a minimal clone. Thus p is prime, and we are done. □

Lemma 2. *If $\mathbb{A}$ satisfies $(*)$ and doesn't have any nontrivial term $f \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ which satisfies the identity*

$$f(f(x,y),y) \approx f(x,y),$$

then $\mathbb{A}$ is a p-cyclic groupoid.

Proof. We define a second composition law $*_2$ on $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ by

$$(f *_2 g)(x,y) = g(f(x,y),y).$$

The assumption implies that $G = (\mathrm{Clo}_2^{\pi_1}(\mathbb{A}), *_2)$ is a group.

Now we define an action $\cdot$ of G on $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ by

$$(g \cdot f)(x,y) = f(x, g(y,x)).$$

First we verify that this is an action:

$$\begin{aligned}((g *_2 h) \cdot f)(x,y) &= f(x, (g *_2 h)(y,x)) \\ &= f(x, h(g(y,x),x)) \\ &= (h \cdot f)(x, g(y,x)) \\ &= (g \cdot (h \cdot f))(x,y).\end{aligned}$$

Note that $\{\pi_1\}$ is an orbit of this action, so every nontrivial orbit has size at most $|G| - 1$. Thus, by the orbit-stabilizer theorem there are nontrivial $g \in G$ and $f \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ such that $g \cdot f = f$, i.e.

$$f(x, g(y,x)) \approx f(x,y).$$

Now we can apply Lemma 1 to finish the argument. □

Lemma 3. *There is no $\mathbb{A}$ which satisfies $(*)$ and has a nontrivial term $f \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ which satisfies the identity*

$$f(f(x,y),y) \approx f(x,y).$$

Proof. Suppose for contradiction that such $\mathbb{A}$ and f existed. Note that such an $\mathbb{A}$ can't be a p-cyclic groupoid. Define a term t by

$$t(x,y) = f(x, f(y,x)).$$

Now we have

$$\begin{aligned} t(x, f(y,x)) &= f(x, f(f(y,x),x)) \\ &\approx f(x, f(y,x)) \\ &= t(x,y), \end{aligned}$$

so by Lemma 1 t must be trivial. Thus we have

$$f(x, f(y,x)) \approx x,$$

so we have

$$f(f(x,z), f(y,z)) = f(x,z)$$

whenever two of x, y, z are equal, hence always.

Also, by Theorem 4 we have

$$f(x, f(x,y)) \approx x,$$

but these identities imply that

$$\begin{aligned} f(x,y) &\approx f(f(x, f(x,y)), f(y, f(x,y))) \\ &\approx f(x, f(x,y)) \\ &\approx x, \end{aligned}$$

contradicting the assumption that f was nontrivial. □

This finishes up the proof of Theorem 5.

Corollary 1. *If $\mathbb{A}$ is a minimal binary algebra which is not Taylor, not a rectangular band, and has no nontrivial term $g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ satisfying the identity*

$$g(g(x,y),y) \approx g(x,y),$$

then $\mathbb{A}$ is a p-cyclic groupoid.

Proof. We define the composition law $*_2$ on $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ by

$$(f *_2 g)(x,y) = g(f(x,y),y).$$

The assumption implies that $G = (\mathrm{Clo}_2^{\pi_1}(\mathbb{A}), *_2)$ is a group.

We'll show that the free algebra $\mathcal{F}_{\mathbb{A}}(x,y)$ has a large automorphism group: for any $f \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$, the fact that G forms a group implies that x is in the subalgebra generated by $f(x,y)$ and y, so

$$\mathrm{Sg}_{\mathcal{F}_{\mathbb{A}}(x,y)}(f(x,y),y) = \mathcal{F}_{\mathbb{A}}(x,y).$$

Thus the homomorphism $\mathcal{F}_{\mathbb{A}}(x,y) \to \mathcal{F}_{\mathbb{A}}(x,y)$ which sends x to $f(x,y)$ and y to y is surjective, so it must be an automorphism (by the finiteness of $\mathcal{F}_{\mathbb{A}}(x,y)$). Similarly, the map which sends x to x and y to $f(y,x)$ is also an automorphism. Since this holds for any $f \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$, we can compose automorphisms of both types to show that for any $f, g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ the map sending x to $f(x,y)$ and y to $g(y,x)$ is an automorphism.

In particular, for any $f \in \mathrm{Clo}_2(\mathbb{A})$ the map

$$(x,y) \mapsto (f(x,y), f(y,x))$$

is invertible, so f can't satisfy the identity $f(f(x,y), f(y,x)) \approx f(x,y)$ unless f is trivial. □

One application is the following connectivity result.

Definition 14. For any algebra $\mathbb{A}$, we let $\mathcal{G}_{\mathbb{A}}$ be the graph of two element subalgebras of $\mathbb{A}$.

Theorem 6. *If $\mathbb{A}$ is an idempotent minimal clone which is neither an affine algebra over $\mathbb{F}_p$ nor a p-cyclic groupoid, then the graph $\mathcal{G}_{\mathbb{A}}$ is connected.*

Proof. This is easy to check in the cases where $\mathbb{A}$ has a semiprojection or is a rectangular band. If $\mathbb{A}$ is Taylor then we know that it must have a nontrivial partial semilattice operation since it is neither affine nor majority. By Theorem 5, we see that $\mathbb{A}$ has a nontrivial term f satisfying the identity

$$f(f(x,y),f(y,x)) \approx f(x,y).$$

If $\mathbb{A}$ has a nontrivial partial semilattice operation s, then the set $\{x, s(x,y)\}$ is a two element subalgebra of $\mathcal{F}_{\mathbb{A}}(x,y)$. Since $f \in \text{Clo}(s)$, we see that $f(x,y)$ is connected to at least one of x, y in $\mathcal{G}_{\mathbb{F}(x,y)}$, and since $f(x,y)$ is adjacent to $f(y,x)$ we see that x and y must be connected to eachother.

Now suppose that $\mathbb{A}$ has no nontrivial partial semilattice operations. By Corollary 1, $\mathbb{A}$ has a nontrivial $g \in \text{Clo}_2^{\pi_1}(\mathbb{A})$ satisfying

$$g(g(x,y),y) \approx g(x,y).$$

Define a term u by

$$u(x,y) = g(g(x,y),x).$$

Then u is a nontrivial term (by Theorem 4 and the fact that g is nontrivial) satisfying the identity

$$u(u(x,y),x) \approx u(x,y),$$

and by Theorem 4 we have

$$u(x,u(x,y)) \approx x,$$

so $\{x, u(x,y)\}$ is a two element subalgebra of $\mathcal{F}_{\mathbb{A}}(x,y)$. Since $f \in \text{Clo}(u)$, we can finish the proof with the same argument we used in the partial semilattice case. □

Next we introduce a class of binary minimal clones which has a nice structure theory (which I think is new).

Definition 15. We'll call an algebra $\mathbb{A} = (A, f)$ a *meld* if f is idempotent and satisfies the identity

$$f(f(x,y),f(z,x)) \approx f(x,y). \tag{$\mathcal{G}$}$$

Theorem 7. *Suppose $\mathbb{A}$ has no ternary semiprojections and has a nontrivial term f that satisfies the identity*

$$f(x,f(x,y)) \approx f(x,f(y,x)) \approx x.$$

Then f satisfies the identity

$$f(f(x,y),f(z,x)) \approx f(x,y).$$

Any nontrivial idempotent f satisfying the above identity defines a minimal clone.

Proof. The given identity implies that

$$\begin{aligned} f(f(x,y),y) &\approx f(f(x,y),f(y,f(x,y))) \\ &\approx f(x,y) \end{aligned}$$

and

$$\begin{aligned} f(f(x,y),x) &\approx f(f(x,y),f(x,f(x,y))) \\ &\approx f(x,y). \end{aligned}$$

Thus, we have

$$f(x,f(f(y,x),z)) = x$$

whenever two of x, y, z are equal. Since $\mathbb{A}$ has no ternary semiprojections, it must hold identically. Replacing x with $f(x,y)$, we get

$$\begin{aligned} f(x,y) &\approx f(f(x,y),f(f(y,f(x,y)),z)) \\ &\approx f(f(x,y),f(y,z)), \end{aligned}$$

and similarly replacing x with $f(x,y)$ and y with $f(z,x)$, we get

$$\begin{aligned}f(x,y) &\approx f(f(x,y),f(f(f(z,x),f(x,y)),z))\\ &\approx f(f(x,y),f(f(z,x),z))\\ &\approx f(f(x,y),f(z,x)).\end{aligned}$$

Now we prove that such an f defines a minimal clone. Suppose that f is a nontrivial idempotent operation satisfying the identity

$$f(f(x,y),f(z,x)) \approx f(x,y).$$

First we show that whenever we have $f(a,b) = a$, we also have $f(b,a) = b$. Supposing that $f(a,b) = a$, we have

$$f(b,a) = f(f(b,b),f(a,b)) = f(b,b) = b.$$

Therefore the graph $\mathcal{G}_{\mathbb{A}}$ has an edge connecting a to b whenever $f(a,b) = a$.

I claim that for any x,y, $f(x,y)$ is adjacent to x, y, and to every neighbour of x in $\mathcal{G}_{\mathbb{A}}$. To see this, we just check that

$$f(f(x,y),x) \approx f(f(x,y),f(x,x)) \approx f(x,y),$$

that

$$f(y,f(x,y)) \approx f(f(y,y),f(x,y)) \approx f(y,y) \approx y,$$

and that for any z with $f(z,x) = z$ we have

$$f(f(x,y),z) = f(f(x,y),f(z,x)) = f(x,y).$$

Applying this repeatedly, we see that $\mathcal{G}_{\mathbb{A}}$ has a vertex which connects to all other vertices. Since f is nontrivial, we also see that $\mathcal{G}_{\mathbb{A}}$ is not a complete graph. Conversely, for any such graph $\mathcal{G}$ and any idempotent function f which restricts to first projection on every edge of $\mathcal{G}$ and which has the property that $f(x,y)$ connects to x, y, and every neighbour of x, f will satisfy the identity $f(f(x,y),f(z,x)) \approx f(x,y)$.

It's easy to see that the free algebra $\mathcal{F}_{\mathbb{A}}(x,y)$ has exactly four elements ($x, y, f(x,y)$, and $f(y,x)$), and the associated graph has only one non-edge (between x and y). So the only way that $\mathbb{A}$ can fail to be a minimal clone is if it has a semiprojection. I claim that if $t \in \mathrm{Clo}(f)$ has

$$t(x,y,...,y) \approx x,$$

then t is first projection, and that if

$$t(x,y,...,y) \not\approx y$$

then $t(x,y,z,...)$ must be adjacent to x in the graph associated to $\mathcal{F}_{\mathbb{A}}(x,y,z,...)$. These can be proved easily by an induction on the size of the definition of t (in terms of f) together with our understanding of the structure of $\mathcal{F}_{\mathbb{A}}(x,y)$. □

Proposition 16. *If $\mathbb{A}$ is a minimal clone and any nontrivial $\mathbb{B} \in \mathrm{Var}(\mathbb{A})$ is a meld, then so is $\mathbb{A}$.*

Lemma 4. *If $\mathbb{A}$ is a binary minimal clone, not a rectangular band, such that there are nontrivial terms $f, g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ satisfying the identity*

$$f(x,g(y,x)) \approx x,$$

then $\mathbb{A}$ is a meld.

Proof. Suppose $\mathbb{A}$ is a counterexample of minimal size (so, in particular, every proper subalgebra or quotient of $\mathbb{A}$ is a set). The argument of Theorem 4 shows that $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$, considered as a subalgebra of $\mathbb{F}_{\mathbb{A}}(x,y)$, is a set.

We will first prove that for any $n \ge 0$ and any $z_1,...,z_n$, we have

$$f(x,g(...g(g(y,x),z_1),...,z_n)) \approx x.$$

We prove this by induction on n. Since $\mathbb{A}$ has no semiprojections, we just have to check the above identity when at most two distinct values occur among $x, y, z_1, ..., z_n$. If $x = y$, the identity follows from the fact that $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ is a set. If $z_n = x$, this follows from the identity $f(x, g(..., x)) \approx x$. If $z_n = y$, then from the fact that $\mathrm{Clo}_2^{\pi_1}$ is a set, we have

$$f(x, g(g(...g(g(y, x), z_1), ..., z_{n-1}), y)) \approx f(x, g(...g(g(y, x), z_1), ..., z_{n-1})),$$

and the claim follows from the inductive hypothesis.

Since g generates $\mathrm{Clo}(\mathbb{A})$, we see that for any $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$, we have

$$f(x, h(g(y, x), x)) \approx x.$$

Since $\mathbb{A}$ does not satisfy the conditions of Theorem 7, there are $a, b \in \mathbb{A}$ with

$$g(a, g(b, a)) \neq a.$$

Thus a and $g(b, a)$ generate $\mathbb{A}$. For any $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$, we then have

$$f(a, h(g(b, a), a)) = a = f(a, h(a, g(b, a))),$$

so for all $c \in \mathbb{A}$ we have $f(a, c) = a$. Since $g \in \mathrm{Clo}_2^{\pi_1}(f)$, this implies that for all $c \in \mathbb{A}$ we have $g(a, c) = a$, but taking $c = g(b, a)$ gives us a contradiction. □

Lemma 5. *If $\mathbb{A}$ is a binary minimal clone, not a rectangular band, such that there are $f, g, h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ with f, g nontrivial satisfying the identities*

$$f(h(x, y), g(y, x)) \approx x,$$
$$g(g(x, y), y) \approx g(x, y),$$

then $\mathbb{A}$ is a meld.

Proof. Again, we take $\mathbb{A}$ to be a minimal counterexample, and let $a, b \in \mathbb{A}$ such that

$$g(a, g(b, a)) \neq a.$$

Similarly to Lemma 4, we show that

$$f(h(x, y), g(...g(g(y, x), z_1), ..., z_n)) \approx x,$$

this time using the identity $g(g(y, x), x) \approx g(g(y, x), y) \approx g(y, x)$. From this we see that for all $c \in \mathbb{A}$ we have

$$f(h(a, b), c) \in \{a, h(a, b)\}.$$

If $h(a, b) = a$ then we finish as in Lemma 4. Otherwise, the argument of Theorem 4 shows that $\mathbb{A}$ has a congruence $\sim$ such that $\mathbb{A}/\sim$ is a two element set, and that for any $c \sim b$ we have

$$\mathrm{Sg}_{\mathbb{A}}\{h(a, b), c\} = \mathbb{A}.$$

By the condition on g, we see that $\mathbb{A}$ can't be a p-cyclic groupoid, so by Theorem 5 $\mathbb{A}$ has a nontrivial term t satisfying

$$t(t(x, y), t(y, x)) \approx t(x, y).$$

From $h \in \mathrm{Clo}(t)$ and $h(a, b) \neq a, b$, we see that there are $u, v \in \mathbb{A} \setminus \{h(a, b)\}$ with $t(u, v) = h(a, b)$. Since each $\sim$-class of $\mathbb{A}$ is a set, we have $u \not\sim v$, so

$$t(v, u) \sim b.$$

But then we have

$$\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}\{h(a, b), t(v, u)\} = \mathrm{Sg}_{\mathbb{A}}\{t(u, v), t(v, u)\} \neq \mathbb{A},$$

a contradiction. □

Definition 16. We define the variety $\mathcal{D}$ of idempotent groupoids with basic operation f satisfying

$$(\mathcal{D}1) \qquad f(x, f(y, x)) \approx f(f(x, y), x) \approx f(f(x, y), y) \approx f(f(x, y), f(y, x)) \approx f(x, y)$$

and

$$(\mathcal{D}2) \qquad \forall n \geq 0 \quad f(x, f(...f(f(x, y_1), y_2), ..., y_n)) \approx x.$$

We say that a binary algebra $\mathbb{A} = (A, f)$ is *weakly dispersive* if there is a surjective map

$$\mathcal{F}_{\mathbb{A}}(x, y) \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x, y)$$

and if f satisfies $(\mathcal{D}2)$.

Note that $\mathcal{F}_{\mathcal{D}}(x, y)$ is a four element algebra, with the following multiplication table.

$\mathcal{F}_{\mathcal{D}}(x, y)$	x	y	$f(x, y)$	$f(y, x)$
x	x	$f(x, y)$	x	$f(x, y)$
y	$f(y, x)$	y	$f(y, x)$	y
$f(x, y)$	$f(x, y)$	$f(x, y)$	$f(x, y)$	$f(x, y)$
$f(y, x)$	$f(y, x)$	$f(y, x)$	$f(y, x)$	$f(y, x)$

Proposition 17. *If $\mathbb{A}$ is a weakly dispersive algebra, then* $\text{Clo}(\mathbb{A})$ *has no semiprojections. If $g \in \text{Clo}(\mathbb{A})$ is nontrivial, then there is a nontrivial $f' \in \text{Clo}(g)$ which is weakly dispersive.*

Proof. First, we prove that the preimage of x in the homomorphism $\mathcal{F}_{\mathbb{A}}(x, y) \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x, y)$ is $\{x\}$ (by induction on the definition of any $g \in \text{Clo}_2(\mathbb{A})$ in terms of f, using the explicit description of $\mathcal{F}_{\mathcal{D}}(x, y)$ as a four element algebra).

Second, we use this to show that any term $t \in \text{Clo}(\mathbb{A})$ which satisfies the identity

$$t(x, y, ..., y) \approx x$$

is first projection (using another induction on the definition of t in terms of f, together with $(\mathcal{D}2)$).

Finally, if $g \in \text{Clo}(\mathbb{A})$ is nontrivial then since g is not a semiprojection and $\mathbb{A}$ is not Taylor, we can find a nontrivial $f' \in \text{Clo}_2^{\pi_1}(g)$ by permuting and identifying inputs of g. Then by the first step the image of f' in $\mathcal{F}_{\mathcal{D}}(x, y)$ is nontrivial, so the map $(\text{Clo}_2(f'), f') \to \mathcal{F}_{\mathcal{D}}(x, y)$ is surjective (and can be checked to be a homomorphism). Since $f' \in \text{Clo}_2^{\pi_1}(f)$, f' satisfies $(\mathcal{D}2)$. □

Theorem 8. *If $\mathbb{A}$ is a minimal binary clone which is not Taylor, has no partial semilattice operations, is not a rectangular band or a p-cyclic groupoid, and is not a neghborhood algebra, then $\mathbb{A}$ is a weakly dispersive algebra.*

Proof. By Theorem 4 and the fact that $\mathbb{A}$ has no semiprojections, any binary operation of $\mathbb{A}$ satisfies the identity $(\mathcal{D}2)$. By Corollary 1, $\mathbb{A}$ has a nontrivial term $t \in \text{Clo}_2^{\pi_1}(\mathbb{A})$ satisfying

$$t(t(x, y), y) \approx t(x, y).$$

Suppose first that there are nontrivial $f, g \in \text{Clo}_2^{\pi_1}(\mathbb{A})$ such that

$$f(g(x, y), y) \approx x.$$

Then we have

$$f(g(x, t(y, x)), t(y, x)) \approx x,$$

and taking $h(x, y) = g(x, t(y, x))$ we see that this contradicts Lemma 5.

Now suppose that there are nontrivial $f, g, h \in \text{Clo}_2^{\pi_1}(\mathbb{A})$ such that

$$f(h(x, y), g(y, x)) \approx x.$$

Define $g_n(x, y)$ inductively by $g_0 = \pi_1, g_1 = g$, and

$$g_{n+1}(x, y) = g(g_n(x, y), y).$$

Then for any $n \geq 1$ we have

$$f(h(x, g_{n-1}(y, x)), g_n(y, x)) \approx x.$$

If there is any n such that g_n is trivial, then we must have $h(x, g_{n-1}(y, x))$ trivial (otherwise we are in the case of the previous paragraph), but then we see that $f(x, y) \approx x$, contradicting the assumption that f is nontrivial. Otherwise, there is some n such that $g_n(g_n(x, y), y) \approx g_n(x, y)$, and we get a contradiction to Lemma 5.

By the last two paragraphs, Lemma 4, and Theorem 4, we see that there is a surjective map

$$\mathcal{F}_{\mathbb{A}}(x, y) \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x, y),$$

and this finishes the proof. □

Proposition 18. *If a binary minimal clone $\mathbb{A}$ is a weakly dispersive algebra, then for any $a, b \in \mathbb{A}$ which generate a nontrivial subalgebra of $\mathbb{A}$ there is a surjective homomorphism*

$$\mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x, y).$$

Definition 17. We say an algebra $\mathbb{A} = (A, f)$ is *dispersive* if f satisfies $(\mathcal{D}2)$, and if for any $a, b \in \mathbb{A}$ which generate a nontrivial subalgebra of $\mathbb{A}$ there is a surjective homomorphism $\mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x, y)$.

Proposition 19. *If $\mathbb{A} = (A, f)$ is a dispersive algebra, then for any $a \neq b \in \mathbb{A}$ and any $c \in \mathbb{A}$, we have $(c, c) \notin \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$. In particular, we have $f(a, b) \neq f(b, a)$.*

Definition 18. If $\mathbb{A} = (A, f)$ is a dispersive algebra, we define the digraph $D_{\mathbb{A}}$ to be the set of edges from a to $f(a, b)$ for all pairs $a, b \in \mathbb{A}$ with $f(a, b) \neq a$. We think of this as a labeled digraph, where the label of an edge $a \to c$ is the set of all elements b such that $f(a, b) = c$. We define the *right orbit* of a, written $\hat{a}$, to be the set of elements which are reachable from a in $D_{\mathbb{A}}$. We define the preorder $\preceq$ on $\mathbb{A}$ by $a \preceq b$ iff $b \in \hat{a}$, and we consider $\preceq$ to be a partial order on the strongly connected components of $D_{\mathbb{A}}$ in the obvious way. We say that a subalgebra $\mathbb{B} \subseteq \mathbb{A}$ is a *strong subalgebra* of $\mathbb{A}$ if for all $b \in \mathbb{B}, a \in \mathbb{A}$, we have $f(b, a) \in \mathbb{B}$.

Proposition 20. *If $\mathbb{A}$ is a dispersive algebra, then every strongly connected component of $D_{\mathbb{A}}$ is a set subalgebra of $\mathbb{A}$.*

Proposition 21. *If $\mathbb{A} = (A, f) = \mathrm{Sg}\{a, b\}$ is a dispersive algebra with $a \neq b$, then $b \notin \hat{a}$, that is, $a \not\preceq b$. In particular, $f(a, b) \neq b$.*

Proposition 22. *If $\mathbb{A}$ is a minimal dispersive algebra, then the set of strongly connected components of $D_{\mathbb{A}}$ and the partial ordering $\preceq$ on them depends only on the clone of $\mathbb{A}$, that is, for any $g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$, the digraph $D_{(A,g)}$ has the same collection of strongly connected components and the same partial ordering $\preceq$. Equivalently, any strong subalgebra of $\mathbb{A}$ is also a strong subalgebra of (A, g).*

Proposition 23. *For any dispersive algebra $\mathbb{A} = (A, f)$, there is a term $g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ such that as elements of $\mathcal{F}_{\mathbb{A}}(x, y)$ we have $f(x, y) \preceq g(x, y)$ and $g(x, y)$ contained in a maximal strongly connected component of $\mathcal{F}_{\mathbb{A}}(x, y)$, and such that g satisfies any* one *of the identities $g(g(x, y), y) \approx g(x, y)$, $g(g(x, y), g(y, x)) \approx g(x, y)$, $g(g(x, y), x) \approx g(x, y)$.*

Definition 19. For every $n \geq 1$, we define the minimal dispersive algebras $\mathbb{L}_n, \mathbb{L}'_n$ on the set $\{a, b, c_0, ..., c_{n-1}, d_0, ..., d_{n-1}\}$ by the multiplication tables

$\mathbb{L}_n$	a	b	c_j	d_j
a	a	c_0	a	a
b	b	b	b	b
c_i	d_i	c_i	c_i	c_i
d_i	d_i	c_{i+1}	d_i	d_i

$\mathbb{L}'_n$	a	b	c_j	d_j
a	a	d_0	a	a
b	b	b	b	b
c_i	d_i	c_i	c_i	c_i
d_i	d_i	c_{i+1}	d_i	d_i

where the indices are considered cyclically modulo n.

Note that the algebras $\mathbb{L}_n, \mathbb{L}_n'$ are term-equivalent. The basic operation of $\mathbb{L}_n$ satisfies the identity $f(f(x,y),y) \approx f(x,y) \approx f(f(x,y),f(y,x))$, while the basic operation of $\mathbb{L}_n'$ satisfes the identity $f(f(x,y),x) \approx f(x,y)$, but no nontrivial term of either algebra satisfies both of these identities simultaneously.

Proposition 24. *Suppose that a binary minimal clone $\mathbb{A} = (A, f)$ is a dispersive algebra, with $\mathbb{A} = \mathrm{Sg}\{a,b\}$ such that for all $c \in \mathbb{A}$, we have $f(b,c) = b$. Then either $|\mathbb{A}| \le 3$, or $\mathbb{A}$ is isomorphic to one of the algebras $\mathbb{L}_n, \mathbb{L}_n'$ for some $n \ge 1$.*

Proof. Choose $g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ such that $g(g(x,y),y) \approx g(x,y)$ and such that $g(a,b)$ is contained in a maximal strongly connected component of $\mathbb{A}$. Let $c = g(a,b)$. Then since $a \preceq c$, we have $g(a,x) = a$ for all $x \in \hat{c}$, so $\hat{c} \cup \{a,b\}$ is closed under g and must therefore be equal to $\mathbb{A}$. Furthermore, since $\hat{c}$ is strongly connected it must be a set. If we consider the digraph $D_{(A,g)}$, we see that the edge labels are subsets of $\{a,b\}$, and that no two consecutive edges can share a label. Assume now that $|\mathbb{A}| > 3$, so $|\hat{c}| \ge 2$. Since every element of $\hat{c}$ has indegree and outdegree at least one, we see that no edge in $\hat{c}$ is labelled $\{a,b\}$, and that in fact each element of $\hat{c}$ must have outdegree exactly one. Thus $\hat{c}$ is a directed cycle, with edge labels alternating between $\{a\}$ and $\{b\}$. ☐

Corollary 2. *If $\mathbb{A}$ is a minimal dispersive algebra and $b \preceq a$ in $\mathbb{A}$, then $\mathrm{Sg}_{\mathbb{A}}\{a,b\}$ either has size at most 3, or is isomorphic to one of the algebras $\mathbb{L}_n, \mathbb{L}_n'$ for some $n \ge 1$.*

Definition 20. We say that a dispersive algebra $\mathbb{A}$ is *inert* if $\hat{a}$ is a set for each $a \in \mathbb{A}$.

Proposition 25. *If $\mathbb{A}$ is a dispersive algebra, then the union of all of the maximal strongly connected components of $D_{\mathbb{A}}$ forms an inert subalgebra of $\mathbb{A}$.*

Proposition 26. *If $\mathbb{A}$ is a minimal inert dispersive algebra, $\mathbb{A} = \mathrm{Sg}\{a,b\}$, and $|\hat{b}| \le 2$, then $|\hat{a}| \le 2$ and $\mathbb{A}$ is isomorphic to a subalgebra of $\mathcal{F}_{\mathcal{D}}(x,y)$.*

Definition 21. For every $n \ge 1$ and every $0 \le k < n$, we define the minimal inert dispersive algebras $\mathbb{W}_{n,k}^a, \mathbb{W}_n^s$ on the set $\{a, b, c_0, ..., c_{n-1}, d_0, ..., d_{n-1}\}$ by the multiplication tables

$\mathbb{W}_{n,k}^a$	a	b	c_j	d_j
a	a	c_0	a	c_{j+1}
b	d_k	b	d_j	b
c_i	c_i	c_i	c_i	c_i
d_i	d_i	d_i	d_i	d_i

$\mathbb{W}_n^s$	a	b	c_j	d_j
a	a	c_0	a	c_{j+1}
b	d_0	b	d_{j+1}	b
c_i	c_i	c_i	c_i	c_i
d_i	d_i	d_i	d_i	d_i

where the indices are considered cyclically modulo n. We define $\mathbb{W}_n^a$ to be $\mathbb{W}_{n,0}^a$.

Proposition 27. *$\mathbb{W}_{n,k}^a$ is isomorphic to $\mathbb{W}_{n,n-1-k}^a$. If n is odd, then $\mathbb{W}_n^s$ is isomorphic to $\mathbb{W}_{n,\frac{n-1}{2}}^a$. For every $n \ge 2$ and every k, $\mathrm{Sg}_{(\mathbb{W}_{n,k}^a)^2}\{(a,b),(b,a)\}$ is isomorphic to $\mathbb{W}_{2n}^s$. Finally, if $\mathbb{A} \in \{\mathbb{W}_{n,k}^a, \mathbb{W}_n^s\}$, then any nontrivial term $g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ defines an algebra which is isomorphic to $\mathbb{A}$.*

Proposition 28. *Up to isomorphism, the only minimal inert dispersive algebras generated by two elements with size ≤ 6 are $\mathcal{F}_{\mathcal{D}}(x,y)$, its three element subalgebra $\{x, f(x,y), f(y,x)\}$, $\mathbb{W}_2^a$, and $\mathbb{W}_2^s$.*

Conjecture 1. If $\mathbb{A}$ is a minimal dispersive algebra, then for any $a \ne b \in \mathbb{A}$, there is a surjective homomorphism from $\mathrm{Sg}\{a,b\}$ to a two element set. Equivalently, the digraph $D_{\mathrm{Sg}\{a,b\}}$ has exactly two weakly connected components.

Conjecture 2. If $\mathbb{A}$ is a minimal dispersive algebra, and if there is some $c \in \mathrm{Sg}\{a,b\}$ such that $\{c,d\}$ is a two element set for all $d \in \mathrm{Sg}\{a,b\}$, then $\mathrm{Sg}\{a,b\} = \{a,b,c\}$. It's enough to show that at least one of $(a,c),(b,c)$ is contained in $\mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$.

Conjecture 3. If $\mathbb{A}$ is a minimal dispersive algebra which is generated by two elements, then $D_{\mathbb{A}}$ has at least three strongly connected components.

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