Coarse Classification of Binary Minimal Clones

Zarathustra Brady

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- $\mathbb{A}$ is called a *set* if all of its operations are projections. Otherwise, we say $\mathbb{A}$ is *nontrivial*.
- If $\text{Clo}(\mathbb{A})$ is minimal and $\mathbb{B} \in \text{Var}(\mathbb{A})$ nontrivial, then $\text{Clo}(\mathbb{B})$ is minimal.

Rosenberg's Five Types Theorem

Theorem (Rosenberg)

Suppose that $\mathbb{A} = (A, f)$ is a finite clone-minimal algebra, and f has minimal arity among nontrivial elements of $\mathrm{Clo}(\mathbb{A})$. Then one of the following is true:

1. *f is a unary operation which is either a permutation of prime order or satisfies $f(f(x)) \approx f(x)$,*
2. *f is ternary, and $\mathbb{A}$ is the idempotent reduct of a vector space over $\mathbb{F}_2$,*
3. *f is a ternary majority operation,*
4. *f is a semiprojection of arity at least 3,*
5. *f is an idempotent binary operation.*

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- The first four cases in Rosenberg's classification are nice properties.

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  - $\implies g$ has a majority term as an identification minor.

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Coarse Classification

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- We'll call such a list a *coarse classification* of minimal clones.
- By Rosenberg's result, we just need to find a coarse classification of *binary* minimal clones.
- The main challenge is to find properties of binary operations f that ensure that $\mathrm{Clo}(f)$ doesn't contain any semiprojections.

Taylor Case

- Theorem (Z.)
  Suppose $\mathbb{A}$ *is a finite algebra which is both clone-minimal and Taylor. Then one of the following is true:*

  1. $\mathbb{A}$ *is the idempotent reduct of a vector space over* $\mathbb{F}_p$ *for some prime* p*,*
  2. $\mathbb{A}$ *is a majority algebra,*
  3. $\mathbb{A}$ *is a spiral.*

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  - The proof uses the characterization of bounded width algebras.
  - All three cases are given by nice properties.

Spirals

- Definition
  $\mathbb{A} = (A, f)$ is a spiral if f is binary, idempotent, commutative, and for any $a, b \in \mathbb{A}$ either $\{a, b\}$ is a subalgebra of $\mathbb{A}$, or $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$ has a surjective map to the free semilattice on two generators.

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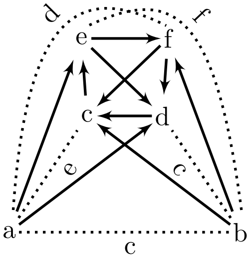
  - Any 2-semilattice is a (clone-minimal) spiral.

  - A clone-minimal spiral which is not a 2-semilattice:



f	a	b	c	d	e	f
a	a	c	e	d	e	d
b	c	b	c	c	f	f
c	e	c	c	c	e	c
d	d	c	c	d	d	d
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Suppose that $\mathbb{A} = (A, f)$ *is a binary minimal clone which is not Taylor. Then, after possibly replacing* $f(x, y)$ *by* $f(y, x)$*, one of the following is true:*

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1. $\mathbb{A}$ *is a rectangular band,*
2. *there is a nontrivial* $s \in \mathrm{Clo}(f)$ *which is a "partial semilattice operation":* $s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y)$*,*

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3. *$\mathbb{A}$ is a p-cyclic groupoid for some prime p,*

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3. $\mathbb{A}$ *is a p-cyclic groupoid for some prime p,*
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3. *$\mathbb{A}$ is a p-cyclic groupoid for some prime p,*
4. *$\mathbb{A}$ is an idempotent groupoid satisfying $(xy)(zx) \approx xy$ ("neighborhood algebra"),*
5. *$\mathbb{A}$ is a "dispersive algebra".*

Dispersive algebras: definition

- We define the variety $\mathcal{D}$ of idempotent groupoids satisfying

$$x(yx) \approx (xy)x \approx (xy)y \approx (xy)(yx) \approx xy, \qquad (\mathcal{D}1)$$

$$\forall n \geq 0 \quad x(\ldots((xy_1)y_2)\cdots y_n)) \approx x. \qquad (\mathcal{D}2)$$

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- Proposition (Lévai, Pálfy)

  If $\mathbb{A} \in \mathcal{D}$, *then* $\mathrm{Clo}(\mathbb{A})$ *is a minimal clone. Also,* $\mathcal{F}_{\mathcal{D}}(x, y)$ *has exactly four elements:* x, y, xy, yx.

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- Definition

  An idempotent groupoid $\mathbb{A}$ is *dispersive* if it satisfies $(\mathcal{D}2)$ and if for all $a, b \in \mathbb{A}$, either $\{a, b\}$ is a two element subalgebra of $\mathbb{A}$ or there is a surjective map

$$\mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x, y).$$

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- In the partial semilattice case, there are no absorption identities at all (aside from idempotence).
- The dispersive case can alternatively be described as the case where every absorption identity follows from $(\mathcal{D}2)$:

$$\forall n \geq 0 \quad x(...((xy_1)y_2)\cdots y_n)) \approx x.$$

I call it "dispersive" because there is very little absorption.

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- Proof sketch: Let $t(a, b) = t(b, a) = b$, then take

$$\begin{aligned} t^{n+1}(x, y) &:= t(x, t^n(x, y)), \\ t^\infty(x, y) &:= \lim_{n \to \infty} t^{n!}(x, y), \\ u(x, y) &:= t^\infty(x, t^\infty(y, x)), \\ s(x, y) &:= u^\infty(x, y). \end{aligned}$$

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- The following absorption identities hold on $\mathbb{B}$:

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- Take $u = f(x, w)$, get

$$f(f(x, y), f(z, w)) \approx f(x, w),$$

so $\mathbb{A}$ is a rectangular band.

$\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$

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- There is a unique surjection from $\mathcal{F}_{\mathbb{A}}(x, y)$ onto a two-element set, and $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ is one of the congruence classes of the kernel.

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- From here on, every function we name will always be assumed to be an element of $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$.

Crucial lemma

- Lemma
  Suppose $\mathbb{A}$ *is a binary minimal clone, not Taylor, not a rectangular band, and not a partial semilattice. Then for any* $f, g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$*, we have*
  $$f(x, g(x, y)) \approx x.$$

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  - If $f(a, g(a, b)) \neq a$, then $a, g(a, b)$ must generate $\mathbb{A}$, so there is $h \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ such that $h(a, b) = b$.

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  - Consider the relation $\mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$: either it's the graph of an automorphism, or it has a nontrivial linking congruence, or it's linked.
  - If it's linked, then there is $\mathbb{B} < \mathbb{A}$ such that $\mathbb{B} \times \mathbb{A} \cap \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$ is subdirect... from here it's easy.

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  - $f, g \mapsto f(g(x, y), g(y, x))$.
- The first one is boring by the Lemma.
- What happens if one of the other two operations forms a group on $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$?

Groupy case - continued

- If the operation $f, g \mapsto f(g(x, y), y)$ forms a group on $\mathrm{Clo}_2^{\pi_1}(\mathbb{A})$, then we can use orbit-stabilizer to find nontrivial $f, g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ such that

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- Together with the Lemma from before, we see that

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whenever two of x, y, z are equal.

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- If f^- is the inverse to f in this group, we get

$$f^-(f(x, g(y, z)), y) = x$$

whenever two of x, y, z are equal. Semiprojection?

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- We have nontrivial $f, g \in \mathrm{Clo}_2^{\pi_1}(\mathbb{A})$ such that $f(x, g(y, z)) \approx f(x, y)$.

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- Since $f \in \mathrm{Clo}(g)$, we have

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- Thus
$$f(f(x,y),z) = f(f(x,z),y)$$
whenever two of x, y, z are equal.

p-cyclic groupoids

- An idempotent groupoid $\mathbb{A}$ is a *p-cyclic groupoid* if it satisfies

$$\begin{aligned} x(yz) &\approx xy, \\ (xy)z &\approx (xz)y, \\ (\cdots((xy)y)\cdots y) &\approx x, \end{aligned}$$

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- Theorem (Z.)

If a binary minimal clone is not a rectangular band and does not have any nontrivial term f satisfying the identity

$$f(f(x,y),y) \approx f(x,y),$$

then it is a p-cyclic groupoid for some prime p. (And similarly if there is no $f(f(x,y),f(y,x)) \approx f(x,y)$.)

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- The v_{ij} must satisfy $v_{ii} = 0$, and for any fixed i the set of v_{ij}s have to span A_i.
- The free p-cyclic groupoid on n generators has np^{n-1} elements.

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- Proposition

  If an idempotent groupoid satisfies $x(xy) \approx x(yx) \approx x$ *and has no ternary semiprojections, then it is a neighborhood algebra.*

- Proposition (Lévai, Pálfy)

  Every neighborhood algebra forms a minimal clone.

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- We need to construct a surjection $\mathcal{F}_{\mathbb{A}}(x,y) \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x,y)$.
- The kernel should have equivalence classes $\{x\},\{y\}$, $\mathrm{Clo}_2^{\pi_1}(\mathbb{A}) \setminus \{x\}$, and $\mathrm{Clo}_2^{\pi_2}(\mathbb{A}) \setminus \{y\}$.

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whenever at most two different variables show up on the left hand side. Semiprojection?
- Since we aren't a neighborhood algebra, there must be some a, b such that
$$g(a, g(b, a)) \neq a.$$

Dispersive case - continued

- We have $\mathrm{Sg}_{\mathbb{A}}\{a, g(b, a)\} = \mathbb{A}$ and

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- Since $g \in \mathrm{Clo}(f)$, we get $g(a, g(b,a)) = a$, a contradiction.

Dispersive case - final

- Need to rule out two similar possibilities - the arguments are similar, but now we must use the existence of functions satisfying $f(f(x,y),y) \approx f(x,y)$ or $f(f(x,y),f(y,x)) \approx f(x,y)$.

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- To see that $\mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\} \twoheadrightarrow \mathcal{F}_{\mathcal{D}}(x,y)$ when $\{a,b\}$ is not a subalgebra, note that if $f((a,b),(b,a)) = (a,b)$, then we must have $f(x,y) \approx x$.

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- I don't know if this is true:

Conjecture

If $\mathbb{A}$ is a dispersive binary minimal clone, then for any $a \neq b$ there is a surjective map from $\mathrm{Sg}_{\mathbb{A}}\{a,b\}$ to a two-element set.

Thank you for your attention.