New Sifting Iterations (bringing the combinatorics back)

Zarathustra Brady

Sieve theoretic notation

- If A is a set of integers and $\mathcal{P}$ is a set of primes, then we define

$$\mathcal{S}(A, \mathcal{P}) = \{a \in A \mid \forall p \in \mathcal{P},\ p \nmid a\}.$$

If z is a real number and $\mathcal{P}$ is the set of primes less than z, we abbreviate this to

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- This notation may be abused in various ways.

The dimension of a sieve

- Our running assumption is that there is a real number κ, called the *sifting dimension*, together with a multiplicative function, also called κ by abuse of notation, satisfying $0 \leq \kappa(p) < p$ for all p and

$$\sum_{p \leq x} \kappa(p) \frac{\log(p)}{p} = (\kappa + o(1)) \log(x),$$

and that z, y are such that for every squarefree integer d, all of whose prime factors are less than z, we have

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- This assumption may be weakened to

$$\left| |A_d| - \kappa(d) \frac{y}{d} \right| \leq \kappa(d) \frac{y}{d \log(y/d)^{2\kappa+\epsilon}}$$

  without affecting the quality of sieve-theoretic bounds.

The dimension of a sieve: examples

- If A is an interval of length y, then we can take $\kappa = 1$, and for any d we will have

$$\left| |A_d| - \frac{y}{d} \right| \leq 1.$$

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- If $A = \{n(n+2) \mid n \in [x, x+y)\}$, then $|\mathcal{S}(A, \sqrt{x+y})|$ counts the number of twin primes in the interval $[x, x+y)$. This is a sieve of dimension 2.

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- Counting numbers which can be written as a sum of two squares corresponds to a sieve with $\kappa = \frac{1}{2}$.

Fundamental Lemma of sieve theory

- The naïve approximation, using the Principle of Inclusion and Exclusion:

$$\begin{aligned}\mathcal{S}(A, z) &= \sum_{d \mid \prod_{p<z} p} \mu(d)|A_d| \\ &\approx \sum_{d \mid \prod_{p<z} p} \mu(d)\kappa(d)\frac{y}{d} \\ &= y \prod_{p<z}\left(1 - \frac{\kappa(p)}{p}\right).\end{aligned}$$

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- If $y = z^s$ with s fixed, this is within a constant factor of the truth!

Fundamental Lemma of sieve theory

Lemma (Selberg)

Define functions $f_\kappa(s), F_\kappa(s)$ *with* $f_\kappa(s)$ *as large as possible and* $F_\kappa(s)$ *as small as possible such that if* $y = z^s$ *with* s *fixed and* z *going to infinity, then*

$$f_\kappa(s) + o(1) \leq \frac{\mathcal{S}(A, z)}{y \prod_{p<z} \left(1 - \frac{\kappa(p)}{p}\right)} \leq F_\kappa(s) + o(1)$$

for any weighted set A *satisfying our basic assumption.*

Then the functions $f_\kappa(s), F_\kappa(s)$ *are finite, continuous, monotone, and computable for* $s > 1$*, and they tend to* 1 *exponentially as* s *goes to infinity.*

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- The precise values of f_κ, F_κ are only known in two cases: $\kappa = \frac{1}{2}$ and $\kappa = 1$.

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- The precise values of f_κ, F_κ are only known in two cases: $\kappa = \frac{1}{2}$ and $\kappa = 1$.
- When $\kappa = 1$, writing $f = f_1$ and $F = F_1$, we have

$$\begin{aligned}
F(s) &= \frac{2e^\gamma}{s} && 1 \le s \le 3 \\
\frac{d}{ds}(sF(s)) &= f(s-1) && s \ge 3 \\
f(s) &= \frac{2e^\gamma \log(s-1)}{s} && 2 \le s \le 4 \\
\frac{d}{ds}(sf(s)) &= F(s-1) && s \ge 2
\end{aligned}$$

Sifting Limit

- Often we are interested in proving a nontrivial lower bound on the size of the set $\mathcal{S}(A, z)$ (for instance, we would like to prove that twin primes exist). In other words, we want to show that $f_\kappa(s) > 0$.

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$$\beta_\kappa = \inf\{s \mid f_\kappa(s) > 0\}.$$

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- Selberg: if κ is sufficiently large, then $\beta < 2\kappa + 0.4454$.
- Diamond-Halberstam-Richert: $\beta_{\frac{3}{2}} \leq 3.11582...$, $\beta_2 \leq 4.26645....$

Buchstab iteration

- When $\kappa \leq 1$, the best known sieves are based on Buchstab's identity:
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- This leads to the inequalities
$$s^\kappa f_\kappa(s) \geq s^\kappa - \kappa \int_{t>s} t^{\kappa-1}(F_\kappa(t-1) - 1)dt,$$
$$s^\kappa F_\kappa(s) \leq s^\kappa + \kappa \int_{t>s} t^{\kappa-1}(1 - f_\kappa(t-1))dt.$$

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- Iterative application of these inequalities leads to the β-sieve.
- When κ is $\frac{1}{2}$ or 1, we have equality!

Equality case: the parity problem

- Define weighted sets A^+, A^-, supported on $[1, y]$, so that the weight A^+ assigns to n is $1 - \lambda(n)$ and the weight A^- assigns to n is $1 + \lambda(n)$.

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- These weighted sets satisfy Buchstab-like identities: for any $w \leq z$, we have

$$\mathcal{S}(A^+, z) = \mathcal{S}(A^+, w) - \sum_{w<p<z} \mathcal{S}(A_p^-, p)$$

and

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- For $1 < s < 3$, we have

$$\mathcal{S}(A^+, z) = 2(\pi(y) - \pi(z)) = \frac{2e^\gamma}{s} \frac{y}{e^\gamma \log(z)} + O\left(\frac{y}{\log(z)^2}\right)$$

Equality case: the parity problem

- By iteratively applying the Buchstab-like identities for A^+, A^-, we can inductively prove that

$$\mathcal{S}(A^+, z) = F(s)\frac{y}{e^\gamma \log(z)} + O\left(\frac{y}{\log(z)^2}\right)$$

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- There is a similar construction for $\kappa = \frac{1}{2}$.

New upper bound iteration rule

- Theorem

  For any $w \leq z$, we have

$$\mathcal{S}(A,z) \leq \mathcal{S}(A,w) - \frac{2}{3}\sum_{w \leq p < z} \mathcal{S}(A_p, w) + \frac{1}{3}\sum_{w \leq q < p < z} \mathcal{S}(A_{pq}, w).$$

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- Proof.

$$1 - \frac{2}{3}k + \frac{1}{3}\binom{k}{2} = \left(1 - \frac{k}{2}\right)\left(1 - \frac{k}{3}\right) \geq 0. \qquad \square$$

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  - In practice, the optimal choice of w appears to be $w = \frac{y}{z^{\beta}}$.

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- Corollary

  For any real $t \geq s \geq 2$, we have

$$s^\kappa F_\kappa(s) \leq t^\kappa F_\kappa(t) - \frac{2}{3}\kappa \int\limits_{\frac{1}{t}<x<\frac{1}{s}} t^\kappa f_\kappa(t(1-x))\frac{dx}{x} + \frac{1}{3}\kappa^2 \iint\limits_{\frac{1}{t}<y<x<\frac{1}{s}} t^\kappa F_\kappa(t(1-x-y))\frac{dx}{x}\frac{dy}{y}.$$

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  - Taking $w = \frac{y}{z^\beta}$ corresponds to taking $t = \frac{s}{s-\beta}$.
  - Comparing $t = \frac{s}{s-\beta}$ with the requirement $t \geq s \geq 2$, we see that this upper bound iteration tends to be useful only for $2 \leq s \leq \beta + 1$.

New lower bound iteration rule

- Theorem
  For any $w \le z^2$, we have

$$\mathcal{S}(A,z) \ge \mathcal{S}\left(A,\sqrt{w}\right) - \sum_{\sqrt{w}\le p<z} \mathcal{S}\left(A_p,\frac{w}{p}\right) + \frac{5}{6}\sum_{\frac{w}{p}\le q<p<z} \mathcal{S}\left(A_{pq},\frac{w}{p}\right)$$

$$- \frac{2}{3}\sum_{\substack{\frac{w}{p}\le r<q<p<z \\ qr<w}} \mathcal{S}\left(A_{pqr},\frac{w}{p}\right) - \frac{1}{2}\sum_{\frac{w}{q}\le r<q<p<z} \mathcal{S}\left(A_{pqr},\frac{w}{p}\right).$$

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  - This is loosely based on the identity

$$1-k+\frac{5}{6}\binom{k}{2}-\frac{1}{2}\binom{k}{3} = (1-k)\left(1-\frac{k}{3}\right)\left(1-\frac{k}{4}\right).$$

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  - Again, the optimal choice of w appears to be $w = \frac{y}{z^\beta}$.

New lower bound iteration rule

Corollary

For any real $s \geq t$ with $2t \geq s \geq 3$, we have

$$
\begin{aligned}
s^\kappa f_\kappa(s) \geq (2t)^\kappa f_\kappa(2t) - \kappa & \int\limits_{\frac{1}{2t}<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^\kappa} F_\kappa\left(\frac{1-x}{\frac{1}{t}-x}\right)\frac{dx}{x} \\
& + \frac{5}{6}\kappa^2 \iint\limits_{\frac{1}{t}-x<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^\kappa} f_\kappa\left(\frac{1-x-y}{\frac{1}{t}-x}\right)\frac{dx}{x}\frac{dy}{y} \\
& - \frac{2}{3}\kappa^3 \iiint\limits_{\frac{1}{t}-x<z<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^\kappa} F_\kappa\left(\frac{1-x-y-z}{\frac{1}{t}-x}\right)\frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \\
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\end{aligned}
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Miracle at $\kappa = 1$

- When $\kappa = 1$, if we take $t = \frac{s}{s-2}$, then the new upper bound iteration rule has equality in the range

$$\frac{5}{2} < s < 3,$$

and the new lower bound iteration rule has equality in the range

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- What is going on here?

Miracle at $\kappa = 1$

- In the case of the upper bound iteration, when $\frac{5}{2} < s < 3$ and $t = \frac{s}{s-2}$ we have $3 < t < 5$, so the claimed identity

$$sF(s) = tF(t) - \frac{2}{3} \int\limits_{\frac{1}{t}<x<\frac{1}{s}} tf(t(1-x))\frac{dx}{x}$$

$$+ \frac{1}{3} \iint\limits_{\frac{1}{t}<y<x<\frac{1}{s}} tF(t(1-x-y))\frac{dx}{x}\frac{dy}{y}$$

becomes, using $F(s) = \frac{2e^{\gamma}}{s}$ for $s \leq 3$ and $f(s) = \frac{2e^{\gamma}\log(s-1)}{s}$ for $2 \leq s \leq 4$,

$$1 = \frac{tF(t)}{2e^{\gamma}} - \frac{2}{3} \int\limits_{\frac{1}{t}<x<\frac{1}{s}} \frac{\log(t(1-x))}{1-x}\frac{dx}{x} + \frac{1}{3} \iint\limits_{\frac{1}{t}<y<x<\frac{1}{s}} \frac{1}{1-x-y}\frac{dx}{x}\frac{dy}{y}$$

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- You can check this integral identity by hand, but a similar strategy for the lower bound iteration is hopeless.

The real reason for the miracle

- Recall the equality case sets A^+, A^- have

$$\mathcal{S}(A^+, z) = F(s)\frac{y}{e^\gamma \log(z)} + O\left(\frac{y}{\log(z)^2}\right),$$
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- So to check we have equality in the upper bound sieve iteration, we just need to check that when $z^{\frac{5}{2}} < y < z^3$, we have

$$\mathcal{S}(A^+, z) = \mathcal{S}(A^+, \frac{y}{z^2}) - \frac{2}{3}\sum_{\frac{y}{z^2}\leq p<z} \mathcal{S}(A_p^-, \frac{y}{z^2})$$
$$+ \frac{1}{3}\sum_{\frac{y}{z^2}\leq q<p<z} \mathcal{S}(A_{pq}^+, \frac{y}{z^2}) + O\left(\frac{y}{\log(z)^2}\right).$$

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$$\mathcal{S}(A^+, z) = \mathcal{S}(A^+, \frac{y}{z^2}) - \frac{2}{3} \sum_{\frac{y}{z^2} \leq p < z} \mathcal{S}(A_p^-, \frac{y}{z^2}) + \frac{1}{3} \sum_{\frac{y}{z^2} \leq q < p < z} \mathcal{S}(A_{pq}^+, \frac{y}{z^2}) + O\left(\frac{y}{\log(z)^2}\right).$$

- Every element of A^+ has an odd number of prime factors, so if $d \in A^+$ is counted more times on the right than the left then d must either be a prime between z and $\frac{y}{z^2}$, be nonsquarefree, or have at least five prime factors, all greater than $\frac{y}{z^2} > z^{\frac{1}{2}}$ (making $d > (z^{\frac{1}{2}})^5 > y$).

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- A similar (but more difficult) analysis shows that the lower bound iteration is also optimal at $\kappa = 1$ when $\frac{7}{2} < s < 4$.

Numerical results at $\kappa = \frac{3}{2}$

- Best previous bound for $\beta_{\frac{3}{2}}$ was given by the Diamond-Halberstam-Richert sieve: $\beta_{\frac{3}{2}} \leq 3.11582\ldots$. This sieve is constructed by applying Buchstab iteration to the Selberg sieve.

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- Applying both iteration rules repeatedly with various choices of the parameters, we get $\beta_{\frac{3}{2}} < 3.11549$.

Thank you for your attention.

Bonus: attaching a probability distribution on the triangle to upper bound sieves which are optimal at $\kappa = 1$

We can write a generic upper bound sieve in the form

$$\mathcal{S}(A,z) \le |A| + \sum_{p<z} \lambda\left(\frac{\log(p)}{\log(y)}\right)|A_p| + \sum_{q<p<z} \lambda\left(\frac{\log(p)}{\log(y)}, \frac{\log(q)}{\log(y)}\right)|A_{pq}| + \cdots$$

where λ (supported on tuples which sum to at most 1) is chosen such that, setting

$$\theta(S) = \sum_{A \subseteq S} \lambda(A),$$

we have $\theta(S) \ge 0$ for every finite (multi-)subset S of the interval $[0,1]$.

In order for this to be an optimal sieve at $\kappa = 1$, we need $\theta(S) = 0$ whenever $|S|$ is odd and the sum of the elements of S is equal to 1.

Bonus: attaching a probability distribution on the triangle to upper bound sieves which are optimal at $\kappa = 1$

We restrict our attention to sets of size 1 and 2, and let $f(x) = \theta(2x), g(x, y) = \theta(2x, 2y)$.

Theorem

Suppose $f : [0, 1] \to \mathbb{R}_{\geq 0}$ and $g : [0, 1]^2 \to \mathbb{R}_{\geq 0}$ are nonnegative functions such that there is some $\epsilon > 0$ with

$$x + y \leq 1 \implies g(x, y) = 0,$$

$$|x+y+z-2| \leq \epsilon \implies f(x)+f(y)+f(z) \leq g(x, y)+g(x, z)+g(y, z)+1,$$

$$x+y+z = 2 \implies f(x)+f(y)+f(z) = g(x, y)+g(x, z)+g(y, z)+1.$$

Then there exists a symmetric probability distribution μ supported on the triangle $\{a, b, c \in [0, 1]^3 \mid a + b + c = 2\}$ with

$$f(x) = \mathbb{P}_{\mu(a,b,c)}[a \leq x], \ g(x, y) = \mathbb{P}_{\mu(a,b,c)}[a \leq x \wedge b \leq y]$$

away from a set of measure 0.

Bonus: attaching a probability distribution on the triangle to upper bound sieves which are optimal at $\kappa = 1$

In this framework:

- The β-sieve corresponds to a probability distribution supported on the center point $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ of the triangle.
- The Selberg sieve corresponds to a uniform probability distribution over the triangle.
- The new upper bound sifting iteration rule corresponds to a probability distribution with mass $\frac{1}{3}$ at each of the vertices $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ of the triangle.

Bonus: a first attempt at a new upper bound sieve for the range $\frac{12}{5} < s < \frac{5}{2}$

If every element of A has size at most $y^{\frac{13}{12}}$ and $z^{\frac{12}{5}} < y < z^{\frac{5}{2}}$:

$$\mathcal{S}(A,z) \le \mathcal{S}(A,\frac{y}{z^2}) - \frac{4}{5}\sum_{\frac{y}{z^2}\le p<\frac{z^3}{y}} \mathcal{S}(A_p,\frac{y}{z^2}) - \frac{2}{3}\sum_{\frac{z^3}{y}\le p<\frac{y^2}{z^4}} \mathcal{S}(A_p,\frac{y}{z^2})$$
$$- \frac{8}{15}\sum_{\frac{y^2}{z^4}\le p<z} \mathcal{S}(A_p,\frac{y}{z^2}) + \frac{3}{5}\sum_{\frac{y}{z^2}\le q<p<\frac{z^3}{y}} \mathcal{S}(A_{pq},\frac{y}{z^2})$$
$$+ \frac{7}{15}\sum_{\frac{y}{z^2}\le q<\frac{z^3}{y}\le p<\frac{y^2}{z^4}} \mathcal{S}(A_{pq},\frac{y}{z^2}) + \frac{1}{3}\sum_{\substack{\frac{y}{z^2}\le q<\frac{z^3}{y}\\ \frac{y^2}{z^4}\le p<z}} \mathcal{S}(A_{pq},\frac{y}{z^2})$$
$$+ \frac{1}{3}\sum_{\frac{z^3}{y}\le q<p<\frac{y^2}{z^4}} \mathcal{S}(A_{pq},\frac{y}{z^2}) + \frac{4}{15}\sum_{\frac{z^3}{y}\le q<\frac{y^2}{z^4}\le p<z} \mathcal{S}(A_{pq},\frac{y}{z^2}) +$$

Bonus: a first attempt at a new upper bound sieve for the range $\frac{12}{5} < s < \frac{5}{2}$ (continued)

$$
\begin{aligned}
&+ \frac{1}{5} \sum_{\frac{y^2}{z^4} \le q < p < z} \mathcal{S}(A_{pq}, \frac{y}{z^2}) - \frac{2}{5} \sum_{\substack{\frac{y}{z^2} \le r < q < p < \frac{z^3}{y} \\ pqr^2 < z^2}} \mathcal{S}(A_{pqr}, \frac{y}{z^2}) \\
&- \frac{4}{15} \sum_{\frac{y}{z^2} \le r < q < \frac{z^3}{y} \le p < \frac{y^2}{z^4}} \left(1 - \frac{3\log(qr)}{8\log(y/p)}\right) \mathcal{S}(A_{pqr}, \frac{y}{z^2}) \\
&+ \frac{1}{5} \sum_{\substack{\frac{y}{z^2} \le s < r < q < p < \frac{z^3}{y} \\ pqr^2 < z^2}} \mathcal{S}(A_{pqr}, \frac{y}{z^2}) \\
&+ \frac{1}{10} \sum_{\frac{y}{z^2} \le s < r < q < \frac{z^3}{y} \le p < \frac{y^2}{z^4}} \left(1 - \frac{\log(qrs)}{\log(y/p)}\right)_+ \mathcal{S}(A_{pqr}, \frac{y}{z^2}).
\end{aligned}
$$